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An Extended Market Model for Credit Derivatives

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Abstract. [1]
In this paper, we present a market model for credit derivatives, built under a standard risk-neutral probability. This is achieved through the introduction of a new class of processes, the Default Accumulator Process, which allows to fill the information gap between forward credit default swap and default time. The simulation framework is detailed in order to assert the tractability of the approach.

1. Introduction.

The market of credit derivatives is fast-expanding, and is on the way to achieve liquidity, almost partly, for its primary instruments, the credit default swaps. They now constitute the reference as hedging products, for the exotic derivatives that have been expanding at the same time. The quotations of credit default swap options starts to bring in information about the volatility and the dynamics of the CDS spreads. In some respect, the credit default swap curve appears as the natural underlying for the credit market, and calls for a corresponding coherent modelling.

The credit modelling is classically divided in two main types of approaches. The first class of models, called structural models, pioneered by Black and Scholes (1973) and Merton (1977) uses the firm value as fundamental variable. The default occurs when the firm value hits a trigger, representing to some extent the value of the outstanding debt. Many developments were then done within this framework, introducing stochastic triggers, or jump diffusions for the latent variable. However, the general principle remains unchanged.

Reduced form models represents the usual alternative. Within this approach, the default intensity process is the fundamental variable, and a canonical construction, presented in [1] for example, allows to build the default time. Various models of this type include [5], [16], [6], [3] and [4]. In comparison with interest rates modelling, they play a similar role than short rate models. Even if the credit equivalent HJM framework has been developped (see [5] and [16]), the positivity condition on the default intensity makes it less tractable than the initial setup developed for interest rates.

The common feature of both modelling approaches with regards to market practices is that the CDS spread is obtained only through the pricing of its given pay-off.

The first attempt to build a modelling framework focused on the CDS spread is due to Schönbucher (2000) in [15], then followed by Jamshidian [10] and Hull and White [9]. As the forward CDS spread represents the fundamental variable, this approach is very close to a market model. However, all constructions are made under a specific probability measure, under which the default event has probability zero. This restricts strongly the field of applications, and only specific single-names products can be priced.

The modelling framework we develop in this paper constitutes a market model for credit derivatives. Using forward CDS spreads as fundamental variables, it is built under a standard risk-neutral probability,

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thus allowing for extensions to multi-names or hybrid products. This is achieved mainly by the introduction of a new class of processes, the Default Accumulator Process, that brings the complementary information to build default information.

Then, the natural credit underlying is directly modelled, which is consistent with the observed market practice. The risk-neutral approach allows to speak of an Extended Market Models, making an explicit reference to previous approaches, built under specific probability measures.

The paper is organized as follows: in the first section, we introduce the main notations, and review some standard results relative to market models. Then, the ECMM specific framework is presented, with a particular focus on the introduced Default Accumulator Process. This section presents the main properties of this class of process, and several useful results are presented. To demonstrate the tractability of the approach, we then enter the field of simulation. Specific issues and corresponding solutions are presented and illustrated with numerical results. This is followed by the description of pricing algorithm, in relation with the most current credit products.

2. Notations and Model Setup.

In this part, we give the basic notations which are used throughout the document and we expose the framework in which the model is set. The results enounced here (as well as their proofs) can be found in Schönbucher(2000).

In what follows, we assume we are given a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}\right)$ where $\mathbb{Q}$ stands for the risk-neutral probability. The quantities which are subject to default risk are denoted with an overbar.

τ represents the default time and we denote by $I(t)$ the associated survival indicator function:

$$I(t) := \mathbf{1}_{\{\tau > t\}}.$$

We define the filtration $\mathbb{H}$ by $\mathcal{H}_t := \sigma\left(\{\tau \leq s\}, s \leq t\right)$ and if W is a $\mathbb{Q}$-Brownian motion, we introduce the filtration $\mathbb{F}^W$ defined by

$$\mathcal{F}_t^W := \sigma\left(W_s, s \leq t\right)$$

We also set $\mathbb{F} := \mathbb{F}^W \vee \mathbb{H}$.

2.1. Bond Prices, Interest Rates and Default Time.

2.1.1. Definitions. We consider payoffs that occur on a discrete set of dates $0 = T_0 < T_1 < \ldots < T_N$. We set $\delta_k = T_{k+1} - T_k$ and $\kappa(t) = \min\{k \,|\, T_k \geq t\}$.

In the rest of the paper we use the standard notations:

- r_t: default-free short interest rate.
- b_t: continuously compounded savings account: $b_t = e^{\int_0^t r_u du}$
- β_t: continuously compounded discount factor: $\beta_t = 1/b_t$.
- $B(t,T)$: price at time t of a default-free zero-coupon bond with maturity T. If $t = T_k$, we note $B(t,T_k) = B_k(t)$.
- $I(t)\overline{B}(t;T)$: price at time t of a defaultable zero-coupon bond with maturity T ($\overline{B}(t,T)$ stands for the pre-default price of the bond). Again, we note $\overline{B}(t,T_k) = \overline{B}_k(t)$.

The default-risk factor at time t for maturity T is defined by

$$D(t,T) = \frac{\overline{B}(t,T)}{B(t,T)}$$

and we note $D_k(t) = D(t, T_k)$.

The different forward rates we use throughout this paper are:

- default-free Libor forward rate at time t for the period $[T_k, T_{k+1}]$:

$$L_k(t) = \frac{1}{\delta_k}\left(\frac{B_k(t)}{B_{k+1}(t)} - 1\right)$$

- defaultable Libor forward rate over $[T_k, T_{k+1}]$:

$$\overline{L}_k(t) = \frac{1}{\delta_k}\left(\frac{\overline{B}_k(t)}{\overline{B}_{k+1}(t)} - 1\right)$$

- linear forward default intensity over $[T_k, T_{k+1}]$

$$H_k(t) = \frac{1}{\delta_k}\left(\frac{D_k(t)}{D_{k+1}(t)} - 1\right)$$

- forward credit spread over $[T_k, T_{k+1}]$

$$S_k(t) = \overline{L}_k(t) - L_k(t)$$

We also introduce the forward defaultable BPV (Basis Point Value) for the period $[T_K, T_N]$:

$$\overline{\mathrm{BPV}}_{T_K, T_N}(t) := \sum_{k=K}^{N-1} \delta_k \overline{B}_{k+1}(t), \quad \text{for } t \leq T_K$$

Finally, defining the process D as $D_t = \mathbf{1}_{\{\tau \leq t\}}$, the intensity of τ is the nonnegative adapted process λ such that

$$M_t := D_t - \int_0^{t \wedge \tau} \lambda_u du$$

is a martingale. In this framework, the survival probability up to time t is given by

$$\mathbb{P}[\tau > t] = \mathbb{E}\left[e^{-\int_0^t \lambda_u du}\right]$$

2.1.2. Useful Relationships. These relations directly stem from the definitions above:

- $S_k(t) = H_k(t)[1 + \delta_k L_k(t)]$ for $t \leq T_k$
- $B_q(t) = B_p(t) \prod_{k=p}^{q-1} (1 + \delta_k L_k(t))^{-1}$ for $0 \leq p < q \leq N$ and $t \leq T_p$.

  In particular,

$$B_q(T_p) = \prod_{k=p}^{q-1} (1 + \delta_k L_k(T_p))^{-1}$$

- $D_q(t) = D_p(t)\prod_{k=p}^{q-1}(1+\delta_k H_k(t))^{-1}$ for $0 \le p < q \le N$ and $t \le T_p$.
If $t = T_p$, we have
$$\overline{B}_q(T_p) = B_q(T_p)\prod_{k=p}^{q-1}(1+\delta_k H_k(T_p))^{-1}$$
- $\delta_k H_k(t)\overline{B}_{k+1}(t) = \overline{B}_k(t)\dfrac{B_{k+1}(t)}{B_k(t)} - \overline{B}_{k+1}(t)$ if $t \le T_k$.
- $\sigma_k^H = \sigma_k^S - \dfrac{\delta_k L_k}{1+\delta_k L_k}\sigma_k^L$
- $\overline{L}_k\sigma_k^{\overline{L}} = \sigma_k^L L_k + \sigma_k^S S_k = (1+\delta_k L_k)H_k\sigma_k^H + (1+\delta_k H_k)L_k\sigma_k^L$

2.1.3. Zero-coupon Bonds Dynamics. In the HJM framework, the absence of arbitrage ensures that the default-free and defaultable forward rates follow the dynamics :

$$\begin{cases} df(t,T) = \sigma^f(t,T)\left(\int_t^T \sigma^f(t,s)ds\right)dt + \sigma^f(t,T)dW_t^{\mathbb{Q}} \\ d\overline{f}(t,T) = \sigma^{\overline{f}}(t,T)\left(\int_t^T \sigma^{\overline{f}}(t,s)ds\right)dt + \sigma^{\overline{f}}(t,T)dW_t^{\mathbb{Q}} \\ \overline{f}(t,t) = \lambda(t) + f(t,t). \end{cases}$$

and thus, the dynamics of the zero-coupon bonds are:

$$\begin{cases} \dfrac{dB(t,T)}{B(t,T)} = r_t dt - \alpha(t,T)dW_t^{\mathbb{Q}} \\ \dfrac{d\overline{B}(t,T)}{\overline{B}(t,T)} = (r_t+\lambda_t)dt - \overline{\alpha}(t,T)dW_t^{\mathbb{Q}} \end{cases}$$

where $\alpha(t,T) = \int_t^T \sigma^f(t,s)ds$ and $\overline{\alpha}(t,T) = \int_t^T \sigma^{\overline{f}}(t,s)ds$.

2.2. Changes of Probabilities.

2.2.1. Forward Measures. Given a time T, the numeraire associated with the T-forward probability $\mathbb{Q}^T$ is the default-free zero-coupon bond $B(.,T)$. In the discrete tenor case, we denote by $\mathbb{Q}^k$ the T_k-forward probability.

The Radon-Nikodym density of the change of probability between $\mathbb{Q}$ and $\mathbb{Q}^T$ is given by

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{\beta(t)B(t,T)}{B(0,T)}$$

and since

$$\frac{B(t,T)}{B(0,T)} = \exp\left[\int_0^t \left(r(s) - \frac{1}{2}\alpha^2(s,T)\right)ds - \int_0^t \alpha(s,T)\,dW_s^{\mathbb{Q}}\right]$$

we have

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \exp\left[-\frac{1}{2}\int_0^t \alpha^2(s,T)\,ds - \int_0^t \alpha(s,T)\,dW_s^{\mathbb{Q}}\right]$$

and Girsanov's theorem ensures that $W^{\mathbb{Q}^T}$, defined by $dW_t^{\mathbb{Q}^T} = dW_t^{\mathbb{Q}} + \alpha(t,T)dt$ is a $\mathbb{Q}^T$-Brownian motion.

The change from $\mathbb{Q}^k$ to $\mathbb{Q}^{k+1}$ is given by:

$$\left.\frac{d\mathbb{Q}^k}{d\mathbb{Q}^{k+1}}\right|_{\mathcal{F}_t} = \frac{B_k(t)/B_{k+1}(t)}{B_k(0)/B_{k+1}(0)} = \frac{1+\delta_k L_k(t)}{1+\delta_k L_k(0)}$$

Proposition 2.1. *The change of drift to reach $\mathbb{Q}^{k+1}$ from $\mathbb{Q}^k$ is obtained through the relation:*

$$dW_t^{\mathbb{Q}^k} = dW_t^{\mathbb{Q}^{k+1}} - \frac{\delta_k L_k(t)}{1+\delta_k L_k(t)}\sigma_k^L dt$$

Proof. We define

$$\rho(t) = \left.\frac{d\mathbb{Q}^k}{d\mathbb{Q}^{k+1}}\right|_{\mathcal{F}_t} = \frac{1+\delta_k L_k(t)}{1+\delta_k L_k(0)}.$$

Since $\delta_k L_k = (B_k - B_{k+1})/B_{k+1}$, L_k is a $\mathbb{Q}^{k+1}$-martingale. Thus the dynamics of L_k under $\mathbb{Q}^{k+1}$ is

$$\frac{dL_k}{L_k} = \sigma_k^L dW^{\mathbb{Q}^{k+1}}.$$

Then

$$\frac{d\rho(t)}{\rho(t)} = \frac{\delta_k dL_k(t)}{1+\delta_k L_k(t)} = \frac{\delta_k L_k(t)\sigma_k^L}{1+\delta_k L_k(t)} dW_t^{\mathbb{Q}^{k+1}}$$

which yields the announced change of drift. ∎

As a corollary, we have the recurrence relationship:

$$\alpha_{k+1}(t) = \alpha_k(t) + \frac{\delta_k L_k(t)}{1+\delta_k L_k(t)}\sigma_k^L, \quad 0 \le t \le T_k \tag{1}$$

The following result, taken from [15], allows to interpret $D_k(t)$ as a survival probability:

Proposition 2.2. *We have $I(t)D_k(t) = \mathbb{Q}^k[\tau > T_k|\mathcal{F}_t]$*

Proof. $I(t)D_k(t)$ is a $\mathbb{Q}^k$-martingale since it is equal to the price of an asset $(I(t)\overline{B}_k(t))$ divided by $B_k(t)$. Thus

$$I(t)D_k(t) = \mathbb{E}^{\mathbb{Q}^k}[I(T_k)D_k(T_k)|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}^k}[I(T_k)|\mathcal{F}_t] = \mathbb{Q}^k[\tau > T_k|\mathcal{F}_t]$$

∎

2.2.2. Survival Measures. The numeraire associated with the T-survival probability $\overline{\mathbb{Q}}^T$ is the defaultable zero-coupon bond with maturity T. When $T = T_k$, we use the notation $\overline{\mathbb{Q}}^k = \overline{\mathbb{Q}}^{T_k}$.

The Radon-Nikodym density of the change of probability between $\mathbb{Q}$ and $\overline{\mathbb{Q}}^T$ is given by

$$\left.\frac{d\overline{\mathbb{Q}}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{\beta(t)I(t)\overline{B}(t,T)}{\overline{B}(0,T)}$$

and the change of drift is given by $dW_t^{\overline{\mathbb{Q}}^T} = dW_t^{\mathbb{Q}} + \overline{\alpha}(t,T)dt$.

The T-survival measure satisfies $\overline{\mathbb{Q}}^T[\tau \le T] = 0$ which justifies the name "survival probability" and proves that $\mathbb{Q}^T$ is not equivalent to $\mathbb{Q}$. Nevertheless, $\mathbb{Q}^T$ is absolutely continuous with respect to $\mathbb{Q}$, which ensures that Girsanov's theorem can still be applied.

The Radon-Nikodym density process for the change from $\overline{\mathbb{Q}}^k$ to $\overline{\mathbb{Q}}^{k+1}$ is:

$$\left.\frac{d\overline{\mathbb{Q}}^k}{d\overline{\mathbb{Q}}^{k+1}}\right|_{\mathcal{F}_t} = \frac{\overline{B}_k(t)/\overline{B}_{k+1}(t)}{\overline{B}_k(0)/\overline{B}_{k+1}(0)} = \frac{1+\delta_k\overline{L}_k(t)}{1+\delta_k\overline{L}_k(0)}$$

so that the drift change is obtained through the recurrence relationship:

$$\overline{\alpha}_{k+1}(t) = \overline{\alpha}_k(t) + \frac{\delta_k\overline{L}_k(t)}{1+\delta_k\overline{L}_k(t)}\sigma_k^{\overline{L}}, \quad 0 \le t \le T_k \tag{2}$$

Now, defining $\alpha^D(t,T)$ as minus the volatility of the process $D(t,T)$, we have $\alpha^D(t,T) = \overline{\alpha}(t,T) - \alpha(t,T)$, which leads to the recurrence relationship:

$$\alpha^D_{k+1}(t) = \alpha^D_k(t) + \frac{\delta_k H_k(t)}{1+\delta_k H_k(t)}\sigma^H_k, \quad 0 \le t \le T_k \tag{3}$$

Using α^D, we can specify the Radon-Nikodym density for the change from $\mathbb{Q}^T$ to $\overline{\mathbb{Q}}^T$:

$$\left.\frac{d\overline{\mathbb{Q}}^T}{d\mathbb{Q}^T}\right|_{\mathcal{F}_t} = \frac{I(t)D(t,T)}{D(0,T)} \quad \text{and} \quad dW_t^{\overline{\mathbb{Q}}^T} = dW_t^{\mathbb{Q}^T} + \alpha^D(t,T)dt$$

2.3. Dynamics of S_k and H_k under $\overline{\mathbb{Q}}^{k+1}$.

The definitions of the forward rate L_k and $\overline{L}_k$ show that they are martingales respectively under $\mathbb{Q}^{k+1}$ and $\overline{\mathbb{Q}}^{k+1}$. If we assume that these rates have lognormal dynamics, we have:

$$\frac{dL_k(t)}{L_k(t)} = \sigma^L_k.dW_t^{\mathbb{Q}^{k+1}} \quad \text{and} \quad \frac{d\overline{L}_k(t)}{\overline{L}_k(t)} = \sigma^{\overline{L}}_k.dW_t^{\overline{\mathbb{Q}}^{k+1}}$$

for some constant vectors σ^L_k and $\sigma^{\overline{L}}_k$.

By definition, $S_k = \overline{L}_k - L_k$. Consequently,

$$dS_k(t) = L_k(t)\sigma^L_k\alpha^D_{k+1}(t)dt + S_k(t)\sigma^S_k dW_t^{\overline{\mathbb{Q}}^{k+1}}$$

Then, differentiating the equality $H_k = S_k/(1+\delta_k L_k)$ yields

$$dH_k(t) = \frac{L_k(t)\sigma^L_k}{1+\delta_k L_k(t)}\left[(1+\delta_k H_k(t))\alpha^D_{k+1} - \delta_k H_k(t)\sigma^H_k\right]dt + H_k(t)\sigma^H_k dW_t^{\overline{\mathbb{Q}}^{k+1}} \tag{4}$$

2.4. Independence Hypothesis.

From now on, the "independence" between default-free interest rates and credit will be defined as the independence (in the mathematical sense) between the random variables H_i and L_j for all $i, j \le N$, under the risk-neutral probability. We thus have the characterization

$$\forall i, j \le N, \sigma^L_i.\sigma^H_j = 0$$

As a corollary, we have

$$\forall i, j \le N, \alpha^D_i.\sigma^L_j = 0$$

Then, the dynamics of S_k and H_k under $\overline{\mathbb{Q}}^{k+1}$ become:

$$\frac{dS_k(t)}{S_k(t)} = \sigma^S_k.dW_t^{\overline{\mathbb{Q}}^{k+1}} \tag{5}$$

and

$$\frac{dH_k(t)}{H_k(t)} = \sigma^H_k.dW_t^{\overline{\mathbb{Q}}^{k+1}}. \tag{6}$$

3. Extended Credit Market Model.

In this section, we present in details the extension of the market model introduced previously. The main issue is to transfer this mono-issuer framework under a non credit-specific probability, which will allow for extensions to multi-issuer or to hybrid products.

Moving into a standard risk-neutral or forward measure raises several issues. The first one is to build the default time information. To achieve this, we introduce a new class of fundamental variables: the Default Accumulator Process. These new variables are strongly linked to both default time and forward CDS spread. The second one is to compute the drift term in the diffusion of forward CDS spread, resulting from the change of probability measure. This aspect will be treated further in the next section.

3.1. Default Accumulator Process.

Definition 3.1. *Default Accumulator Process*

We define the Default Accumulator Process of maturity T as the process $(\varepsilon(t,T))_{t>0}$ given, under the probability $\mathbb{Q}^T$, by :

$$\begin{cases} \varepsilon(t,T) = \varepsilon(0,T)\exp\left(-\frac{1}{2}\int_0^t \alpha^D(s,T)^2\,ds - \int_0^t \alpha^D(s,T)\,dW_s^{\mathbb{Q}^T}\right) \\ \varepsilon(0,T) = D(0,T) \end{cases} \tag{7}$$

The introduction of the Default Accumulator Process becomes quite natural when considering the change of probability measure from $\mathbb{Q}^T$ to $\overline{\mathbb{Q}}^T$, as defined in the previous section (see 2.2.2). Then :

$$\begin{aligned} \left.\frac{d\overline{\mathbb{Q}}^T}{d\mathbb{Q}^T}\right|_{\mathcal{F}_t^W} &= \exp\left(-\int_0^t \lambda_s ds\right)\frac{\overline{B}(t,T)}{\overline{B}(0,T)}\frac{B(0,T)}{B(t,T)} \\ &= \frac{\varepsilon(t,T)}{\varepsilon(0,T)} \end{aligned}$$

which shows that $\varepsilon(t,T)$ plays the role of a default tracker, allowing to switch to a probability measure under which the event of default has probability zero. The term structure $(\varepsilon(t,T))_{T>0}$ will allow to get information on the default event and the default time, directly from variables naturally introduced by the market model approach, and under a non-credit specific probability.

Remark 3.2. *In the particular case of zero interest rates, the Default Accumulator Process becomes:*

$$\begin{aligned} \varepsilon(t,T) &= \exp\left(-\int_0^t \lambda_s ds\right)\frac{\overline{B}(t,T)}{\overline{B}(0,T)} \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T \lambda_s ds\right)\middle|\mathcal{F}_t^W\right] \end{aligned}$$

which is a $\mathbb{Q}$-martingale. The DAP is then the conditional expectation of the hazard process Γ_t, and appears as its term structure extension. In particular, the value at time T becomes:

$$\varepsilon(T,T) = \exp(-\Gamma_T)$$

where $\Gamma_t = \int_0^t \lambda_s ds$. This is thus the natural process to focus on in order to get the default time information.

We now turn to the main properties of the Default Accumulator Process.

Lemma 3.3. *SDE under Spot and Forward Neutral Probabilities*
Under the forward measure $\mathbb{Q}^T$, the Default Accumulator Process follows the SDE given by :

$$\begin{cases} \dfrac{d\varepsilon(t,T)}{\varepsilon(t,T)} = -\alpha^D(t,T)\, dW_t^{\mathbb{Q}^T} \\ \varepsilon(0,T) = D(0,T) \end{cases} \tag{8}$$

Under the risk-neutral measure $\mathbb{Q}$, it follows :

$$\begin{cases} \dfrac{d\varepsilon(t,T)}{\varepsilon(t,T)} = \alpha^D(t,T)\,\alpha(t,T)\, dt - \alpha^D(t,T)\, dW_t^{\mathbb{Q}} \\ \varepsilon(0,T) = D(0,T) \end{cases} \tag{9}$$

Proof. Equation (8) is simply a rewriting of definition (7) in terms of SDE. The second equation is obtained by applying the classical change of probability measure, from the T-forward measure to the risk-neutral measure. ■

Lemma 3.4. *Martingale Property*
The Default Process $\varepsilon(t,T)$ is a $\mathbb{Q}^T$-martingale. Furthermore, under the assumption of independence between credit and interest rates, it is a martingale under $\mathbb{Q}$.

Proof. Direct from equations (7) and (9). ■

From equation (7), we see that the Default Accumulator Process is the exponential martingale of the volatility process of $D(t,T)$. Considering the SDE defined in the first section, we find that :

$$\frac{\varepsilon(t,T)}{D(t,T)} = \exp\left(-\int_0^t \lambda_s ds\right) \tag{10}$$

and in particular, taking $t = T$ leads again to:

$$\varepsilon(T,T) = \exp\left(-\int_0^T \lambda_s ds\right) \tag{11}$$

Lemma 3.5. *Decreasing Term Structure*

$$\forall\ k > j,\ \forall\ t \in [0, T_j],\ \ \varepsilon_k(t) < \varepsilon_j(t)$$

Proof. Starting from (10)

$$\frac{\varepsilon_k(t)}{D_k(t)} = \frac{\varepsilon_j(t)}{D_j(t)} = \exp\left(-\int_0^t \lambda_s ds\right)$$

we have:

$$\frac{\varepsilon_k(t)}{\varepsilon_j(t)} = \frac{D_k(t)}{D_j(t)} = \prod_{i=j}^{k-1} \left(1 + \delta_i H_i(t)\right)$$

and the quantity on the right is always positive as H_i is defined as a non-negative process. ■

The next lemma is a reformulation with the DAP of a classical result. This will show that this process can play a role very close to that played by a numeraire.

Lemma 3.6. *Pricing Rule*
For any $T_i < T_k$, and any process X being $\mathcal{F}_{T_k}^W$-measurable, we have :

$$\begin{aligned} V(T_i) &= \mathbb{E}^{\mathbb{Q}}\left[I(T_k) \exp\left(-\int_{T_i}^{T_k} r_s ds\right) X \,|\mathcal{F}_{T_i}\right] \\ &= I(T_i)\,\varepsilon_i(T_i)^{-1}\, \mathbb{E}^{\mathbb{Q}}\left[\varepsilon_k(T_k) \exp\left(-\int_{T_i}^{T_k} r_s ds\right) X \,\middle|\mathcal{F}_{T_i}^W\right] \end{aligned} \tag{12}$$

Proof. Following [1] for example, we have :

$$\begin{aligned} V(T_i) &= \mathbb{E}^{\mathbb{Q}}\left[I(T_k)\exp\left(-\int_{T_i}^{T_k} r_s ds\right) X \left| \mathcal{F}_{T_i} \right. \right] \\ &= I(T_i)\,\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_{T_i}^{T_k} \lambda_s ds\right) \exp\left(-\int_{T_i}^{T_k} r_s ds\right) X \left| \mathcal{F}^W_{T_i} \right. \right] \end{aligned}$$

Using the terminal condition (11) :

$$V(T_i) = I(T_i)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{\varepsilon_k(T_k)}{\varepsilon_i(T_i)} \exp\left(-\int_{T_i}^{T_k} r_s ds\right) X \left| \mathcal{F}^W_{T_i} \right. \right]$$

and the fact that $\varepsilon_i(T_i)$ is $\mathcal{F}^W_{T_i}$-measurable completes the proof. ∎

The properties detailed above will be very useful all along the development of the ECMM, i.e. for its simulation, and for the pricing of credit derivatives within this framework. The next sections will then make an intensive use of these results.

Before turning to simulation and pricing issues, we will make a rewriting of all useful quantities in terms of the Default Accumulator Process, starting with the linear default intensity. For the sake of simplicity, all processes are implicitly taken at time t.

From (10), we recall that, for any $k = 0, \ldots, N-1$ and $t \leq T_k$:

$$\begin{aligned} \frac{\varepsilon_k}{\varepsilon_{k+1}} &= \frac{D_k}{D_{k+1}} \\ &\triangleq 1 + \delta_k H_k \end{aligned}$$

Then, the linear default intensity has a similar expression with the DAP, as with the Default Factor Process D_k:

$$H_k = \frac{1}{\delta_k}\left(\frac{\varepsilon_k}{\varepsilon_{k+1}} - 1\right)$$

Seemingly, the defaultable zero coupon bond is given by:

$$\frac{\overline{B}_{k+1}}{\overline{B}_k} = \frac{B_{k+1}}{B_k}\frac{\varepsilon_{k+1}}{\varepsilon_k}$$

so that, for $t \leq T_l$ and $t \leq T_k$:

$$\frac{\overline{B}_k}{\overline{B}_l} = \frac{B_k}{B_l}\frac{\varepsilon_k}{\varepsilon_l} \tag{13}$$

We now turn to the forward CDS spread over the period $[T_K, T_N]$, under the hypothesis of independence between default and interest rates. Starting from :

$$\left(\sum_{j=K}^{N-1} \delta_j \overline{B}_{j+1}\right) s_{T_K,T_N} = \sum_{k=K}^{N-1} \delta_k \overline{B}_{k+1} H_k$$

we have, with (13) and using a fixed k_0, such that $t \leq T_{k_0}$:

$$\frac{\overline{B}_{k_0}}{B_{k_0}\varepsilon_{k_0}}\left(\sum_{j=K}^{N-1} \delta_j \varepsilon_{j+1} B_{j+1}\right) s_{T_K,T_N} = \frac{\overline{B}_{k_0}}{B_{k_0}\varepsilon_{k_0}} \sum_{k=K}^{N-1} \varepsilon_{k+1} B_{k+1}\left(\frac{\varepsilon_k}{\varepsilon_{k+1}} - 1\right)$$

which leads to:

$$\left(\sum_{j=K}^{N-1} \delta_j \varepsilon_{j+1} B_{j+1}\right) s_{T_K,T_N}(t) = \sum_{k=K}^{N-1} B_{k+1}(t)\left(\varepsilon_k - \varepsilon_{k+1}\right)$$

$$\left(\sum_{j=K}^{N-1}\delta_j\varepsilon_{j+1}B_{j+1}\right)s_{T_K,T_N} = \varepsilon_K B_{K+1}-\varepsilon_N B_N + \sum_{k=K}^{N-2}\varepsilon_{k+1}\left(B_{k+2}-B_{k+1}\right) \tag{14}$$

which gives the forward spread for period $[T_K, T_N]$ directly from the $(\varepsilon_k)_{k=1,\ldots,N}$.

3.2. Change of Numeraire.

One of the challenges of the definition of the ECMM is to build a coherent, arbitrage free, construction of the drifts terms in the diffusion of the credit spreads under some standard neutral probability. To achieve this, we start by giving more precisely the different changes of probability measure, written in terms of Default Accumulator Process.

Lemma 3.7. *From Forward Survival to Forward Neutral Measure*
The change of probability, from $\mathbb{Q}^j$ to $\overline{\mathbb{Q}}^k$, for any k and j, is given in terms of Default Accumulator Process, as:

$$\left.\frac{d\overline{\mathbb{Q}}^k}{d\mathbb{Q}^j}\right|_{\mathcal{F}_t^W} = \exp\left(-\int_0^t \lambda_s ds\right)\frac{\overline{B}_k(t)}{\overline{B}_k(0)}\frac{B_j(0)}{B_j(t)} \tag{15}$$

$$= \varepsilon_k(t)\frac{B_k(t)}{\overline{B}_k(0)}\frac{B_j(0)}{B_j(t)} \tag{16}$$

In particular, for $j = k$:

$$\left.\frac{d\overline{\mathbb{Q}}^k}{d\mathbb{Q}^k}\right|_{\mathcal{F}_t^W} = \frac{\varepsilon_k(t)}{\varepsilon_k(0)}$$

This change of probability measure is valid only for $\mathcal{F}_t^W$-measurable process. It is interesting to note that in this case, the change of probability is standard, giving two equivalent probability measures (see [15] for more details).

Lemma 3.8. *From Forward Defaultable BPV to Forward Neutral Measure*
The change of probability, from $\mathbb{Q}^j$ to $\overline{\mathbb{Q}}^{K,N}$, for any j, K and N, is given in terms of Default Accumulator Process, as:

$$\left.\frac{d\overline{\mathbb{Q}}^{K,N}}{d\mathbb{Q}^j}\right|_{\mathcal{F}_t^W} = \exp\left(-\int_0^t \lambda_s ds\right)\frac{\overline{BPV}_{T_K,T_N}(t)}{\overline{BPV}_{T_K,T_N}(0)}\frac{B_j(t)}{B_j(0)} \tag{17}$$

$$= \frac{\sum_{k=K}^{N}\delta_k B_{k+1}(t)\,\varepsilon_{k+1}(t)}{\sum_{k=K}^{N}\delta_k B_{k+1}(0)\,\varepsilon_{k+1}(0)}\frac{B_j(0)}{B_j(t)}$$

Proof. Using (13), we find:

$$\frac{\overline{\mathrm{BPV}}_{T_K,T_N}(t)}{\overline{B}_N(t)} = \sum_{k=K}^{N}\delta_k B_{k+1}(t)\,\varepsilon_k(t)$$

Rewriting (17) leads to:

$$\left.\frac{d\overline{\mathbb{Q}}^{K,N}}{d\mathbb{Q}^j}\right|_{\mathcal{F}_t^W} = L_j^{K,N}(t)$$

$$= \frac{\overline{\mathrm{BPV}}_{T_K,T_N}(t)}{\overline{\mathrm{BPV}}_{T_K,T_N}(0)}\frac{\overline{B}_N(t)}{\overline{B}_N(0)}\times\exp\left(-\int_0^t \lambda_s ds\right)\frac{B_j(0)}{B_j(t)}\frac{\overline{B}_N(t)}{\overline{B}_N(0)}$$

where the left term is similar to the change of probability defined in (15). Then,

$$\begin{aligned} L_j^{K,N}(t) &= \frac{\sum_{k=K}^{N}\delta_k B_{k+1}(t)\,\varepsilon_k(t)}{\sum_{k=K}^{N}\delta_k B_{k+1}(0)\,\varepsilon_k(0)} \times \frac{B_N(0)\,\varepsilon_N(0)}{B_N(t)\,\varepsilon_N(t)} \times \frac{B_j(0)\,B_N(t)}{B_j(t)\,B_N(0)} \times \varepsilon_N(t) \\ &= \frac{\sum_{k=K}^{N}\delta_k B_{k+1}(t)\,\varepsilon_k(t)}{\sum_{k=K}^{N}\delta_k B_{k+1}(0)\,\varepsilon_k(0)} \frac{B_j(0)}{B_j(0)} D_N(0) \end{aligned}$$

using the fact that $\varepsilon_N(0) = D_N(0)$. ■

3.3. Default Time Definition.

The Default Accumulator Process is now precisely defined, and has been linked to the major quantities introduced for the market model. An interesting property is that, under a well chosen class of probability measures, it represents a family of processes easy to simulate, as being martingales, from which all credit quantities can be recalculated. In some respect, it plays a similar role as the defaultable zero-coupon bond.

However, its most interesting property is that it allows to get the default information. The purpose of this section is to detail the link between the DAP ε and the default time τ. In particular, we will show that the classical definition of τ can be rewritten using the DAP, which implies that the default information is not altered. Then, we introduce another definition for the default time, more coherent with the discrete time framework of the model.

The default time τ is defined classically under $\mathbb{Q}$ as:

$$\tau = \inf\left\{ t > 0 \,/\, \exp\left(-\int_0^t \lambda_s ds \right) < U \right\}$$

where $U \sim \mathcal{U}([0,1])$ and is independent of $\mathbb{F}^W$. Taking directly (11), the definition becomes :

$$\tau = \inf\left\{ t > 0 \,/\, \varepsilon(t,t) < U \right\}$$

or equivalently, for arbitrarily fixed T :

$$\tau = \inf\left\{ t > 0 \,/\, \frac{\varepsilon(t,T)}{D(t,T)} < U \right\} \tag{18}$$

Definition (18) shows that the default time remains unchanged, i.e. that the use of the DAP is only a rewriting, and not a modification of this random variable. As the ECMM framework is set on a discrete schedule, it may however be interesting to introduce another random variable, denoted by $\hat{\tau}$, that would be coherent with this specification.

Definition 3.9. *Default Time Definition*
Given a model schedule $(T_k)_{k=0,\ldots,N}$, we define the related default time as

$$\hat{\tau} = \inf\left\{ T_k \,/\, \varepsilon_k(T_k) < U \right\} \tag{19}$$

where $U \sim \mathcal{U}([0,1])$ and is independent of $\mathbb{F}^W$.

The default time $\hat{\tau}$ can be seen as a restriction of τ on the schedule $(T_k)_{k=0,\ldots,N}$. Note that the property of decreasing term structure guarantees the consistency of this definition.

This completes the general setting of the ECMM, as it introduces a coherent way to compute the default time. Then, according to the previous mechanism, the ECMM is fully specified, under any standard risk-neutral probability, by a class of forward credit spreads and corresponding diffusions.

4. Calibration and Simulation.

The Extended Credit Market Model is fully defined by the joint specification of the diffusion of a well chosen family of forward credit spreads, and by the definition of default time. In this part, we will enter the field of simulation, with a particular focus on default time simulation.

In fact, this represents the main difference between a Credit Market Model and a standard Libor Market Model. It is also a key point in the model, as it is what will allow for an extension to multi-issuers products, for which the pay-offs must be computed directly from default time simulation.

Another remarkable difference, yet easier to deal with, is the fact that the forward spreads diffusions are given under a probability measure that is not natural. This problematic is not specific to credit modelling, as the same issue would arise when considering a LMM under the risk-neutral measure $\mathbb{Q}$. In practice however, the LMM simulation is done under some appropriate forward measure, and a spot martingale measure can also be introduced (cf. [14]). As the credit equivalent probabilities are issuer specific, the same methods cannot be applied in this framework. This means that specific path-dependent drift terms will appear in the diffusion of the forward credit spreads.

This section is composed as follows. We start with the computation of the diffusion of forward credit spreads, under a the forward-neutral probability $\mathbb{Q}^{T_N}$. Then, the specificity and complexity of the drift terms are examined, as we turn to the simulation issues of the ECMM. The section ends with a brief discussion on calibration.

4.1. Model Parametrization.

We introduce different model parametrization, depending on which class of forward CDS spread is chosen.

We denote by $(s_{T_K,T_N}(t))_{t\leq T_K}$ the forward CDS spread for period $[T_K, T_N]$ taken at time t. As done in [15], it is interesting to introduce the survival probability $\overline{\mathbb{Q}}^{K,N}$ associated with the defaultable basis point value for the period $[T_K, T_N]$: $\overline{BPV}_{T_K,T_N}$. As $s_{T_K,T_N}(t)$ is martingale under $\overline{\mathbb{Q}}^{K,N}$, we may write its SDE as:

$$\frac{ds_{T_K,T_N}(t)}{s_{T_K,T_N}(t)} = \sigma_{K,N}.dW_t^{\overline{\mathbb{Q}}^{K,N}} \tag{20}$$

The following definitions make explicit the different parametrizations:

Definition 4.1. *Column Model*

The k_0-column model is represented through the forward CDS spreads: $\left(s_{T_k,T_{k+k_0}}\right)_{k=1,..,N}$

For $k_0 = 1$, we get a model parametrization directly in terms of $(H_k)_{k=1,..,N}$.

Definition 4.2. *Diagonal Model*

The T_N-column model is represented through the forward CDS spreads: $(s_{T_k,T_N})_{k=1,..,N}$

4.2. Forward-Neutral Dynamics.

4.2.1. Linear Forward Default Intensity Process. Considering the change of probability given by (15), we get the corresponding change in Brownian motions:

$$\begin{aligned} dW_t^{\overline{\mathbb{Q}}^k} &= dW_t^{\mathbb{Q}^N} - \left(-\alpha_k^D(t) - \alpha_k(t) + \alpha_N(t)\right)dt \\ &= dW_t^{\mathbb{Q}^N} - \left(\alpha_N(t) - \overline{\alpha}_k(t)\right)dt \end{aligned}$$

Starting from (6) and applying the previous change of probability leads to:

$$dH_k(t) = \frac{L_k(t)\sigma_k^L}{1+\delta_k L_k(t)}\left[(1+\delta_k H_k(t))\alpha_{k+1}^D - \delta_k H_k(t)\sigma_k^H\right]dt + H_k(t)\,\sigma_k^H\left(\overline{\alpha}_k(t) - \alpha_N(t)\right)dt + H_k(t)\,\sigma_k^H dW_t^{\mathbb{Q}^N} \tag{21}$$

4.2.2. Forward CDS Spread. Under the survival probability $\overline{\mathbb{Q}}^{K,N}$, the forward CDS spread for period $[T_K, T_N]$ follows the SDE given by:

$$\frac{ds_{T_K,T_N}(t)}{s_{T_K,T_N}(t)} = \sigma_{K,N}.dW_t^{\overline{\mathbb{Q}}^{K,N}}$$

Considering the change of probability given by (17), we get the corresponding change in Brownian motion as:

$$dW_t^{\overline{\mathbb{Q}}^{K,N}} = dW_t^{\mathbb{Q}^N} - \left(\alpha_N(t) - \frac{\sum_{k=K}^{N-1} \delta_k \overline{B}_{k+1}(t)\,\overline{\alpha}_{k+1}(t)}{\overline{\mathrm{BPV}}_{T_K,T_N}(t)}\right) dt$$

so that:

$$\frac{ds_{T_K,T_N}(t)}{s_{T_K,T_N}(t)} = \sigma_{K,N}\left(\frac{\sum_{k=K}^{N-1} \delta_k \overline{B}_{k+1}(t)\,\overline{\alpha}_{k+1}(t)}{\overline{\mathrm{BPV}}_{T_K,T_N}(t)} - \alpha_N(t)\right) dt + \sigma_{K,N} dW_t^{\mathbb{Q}^N} \tag{22}$$

From both diffusion equations, we see that the fundamental variables involved in the drift term are the processes $(\overline{\alpha}_k)_k$. As said previously, these processes follows a recurrence relationship allowing for an easy computation. Then, the simulation may be done using standard log-Euler scheme, and building step by step the drift term along this recurrence equation.

However, the drift recursion must be initialized, which an important issue in the simulation process. This point is addressed in the next paragraph.

4.3. Drift Complexity and Simulation Restriction.

The drift complexity comes essentially from the initialization of the recurrence relationship for $\overline{\alpha}_k$. However, as $\overline{\alpha}_k = \alpha_k^D + \alpha_k$, and as α_k may be computed according to the recurrence relationship (1), we will focus on the credit specific term α_k^D.

We start with:

$$\begin{aligned}
\alpha_{k+1}^D(t) &= \int_0^{T_{k+1}} \left(\sigma^{\overline{f}}(t,s) - \sigma^f(t,s)\right) ds \\
&= \int_0^{T_{\kappa(t)}} \left(\sigma^{\overline{f}}(t,s) - \sigma^f(t,s)\right) ds + \int_{T_{\kappa(t)}}^{T_{k+1}} \left(\sigma^{\overline{f}}(t,s) - \sigma^f(t,s)\right) ds
\end{aligned}$$

Then, using (3), we get:

$$\alpha_{k+1}^D(t) = \int_0^{T_{\kappa(t)}} \left(\sigma^{\overline{f}}(t,s) - \sigma^f(t,s)\right) ds + \sum_{j=\kappa(t)}^{k+1} \frac{\delta_j H_j(t)\,\sigma_j^H}{1+\delta_j H_j(t)} \tag{23}$$

From this equation, we see that an additional term is needed when t is not one of the T_k:

$$\int_0^{T_{\kappa(t)}} \left(\sigma^{\overline{f}}(t,s) - \sigma^f(t,s)\right) ds$$

However, it is not possible, from given diffusions on forward CDS spreads, to compute this term, as it involves explicitly the volatility of the defaultable forward rates. This problem is very similar to the one that motivates the introduction of a spot martingale measure for the Libor Market Model (see [14]).

This issue implies some numerical constraints for the simulation of the ECMM. Of course, one solution may be to use an interpolation method for simulation dates $t \in]T_k, T_{k+1}[$. But this may have a direct impact on the distribution of the default time τ. Thus, this may be used with precaution.

We propose a different approach. The simulation ECMM is fundamentally linked to the specification of a schedule $(T_k)_{k=0,..,N}$. This schedule can be chosen without any specific constraints, apart from those relative to the forward credit spreads defined as fundamental variables. Then, all the necessary information is reduced to default events on sub-period of time $[T_K, T_{K+1})$. This type of default information clearly involves only simulations of the DAP for dates $(T_k)_{k=0,..,N}$, for which the drift term is known exactly. As a result, we see that the issue of drift initialization can be solved without using any kind of approximation. The weakness of the approach may be a convergence problem for the SDE of the chosen class of forward credit spreads, as the time step in the discretization scheme is imposed. We will see at the end of the section, through a concrete numerical example, that, with the typical schedule used in practice, the simulation is in fact very accurate.

4.4. Spread and Volatility Calibration.

The calibration procedure involves both the initial spread curve and the forward spread volatility term structure. For the sake of simplicity, we take the example of a standard 3-months column model, i.e. a model based on the 3-months forward CDS spreads: $(H_k)_{k=0,\dots,N}$. All numerical examples will be relative to this type of parametrization.

The difficulty of the calibration process depends evidently of the hypothesis chosen for the model. More precisely, the hypothesis of independence between interest rates and credit spreads plays a crucial role. In fact, it allows, as seen in previous sections, to have closed formulas to link fundamental model variables, such as the Default Accumulator Process, or the linear default intensity, to the forward credit spreads. As done by Schönbucher in [15], it is possible to extend these formulas to the case of correlated markets using approximations of the associated convexity corrections. However, recent works from Brigo and Alfonsi (see [4] for example), done within the CIR++ framework, seem to indicate that the effect of interest rate correlation is negligible when pricing a standard credit default swap. We then assume, for the calibration process, that the hypothesis of independence holds. Note that it is possible to proceed to the same process as in [4] within the ECMM.

An important property of the ECMM is that it is auto-calibrated in spread, as the initial market spreads represent simply the initial conditions of the fundamental SDE of the model. A simple recurrence procedure can be used to built the initial term structure $(H_k(0))_{k=0,\dots,N}$ from market prices.

The calibration of volatility may require slightly more efforts, depending on the nature of volatility to be calibrated. However, as shown in ([15]), there exists, in specific cases, closed formulas for Credit Spread Options, allowing for fast calibration. Typically, a Black typed 3-months model is auto-calibrated in volatility on the corresponding 3-months column. However, the calibration of a diagonal within this framework may require a specific procedure.

An interesting feature of the ECMM with respect to calibration is that it is specified under a risk-neutral probability, so that a large part of methods developed for Libor Market Models may be used directly.

Example 4.3. *We consider the case of a 1-column model, i.e.* $k_0 = 1$ *with the notations of definition (4.1.). We chose the issuer France Telecom and we compute its default probabilities with the ECMM for maturities going from 3 months to 10 years.*

The input parameters are the following:

- *for all* k, $\sigma_k^H = \sigma_{k,k+1} = 120\%$,
- *number of Monte Carlo simulations: 100 000*

and the initial spread curve as quoted by the market is:

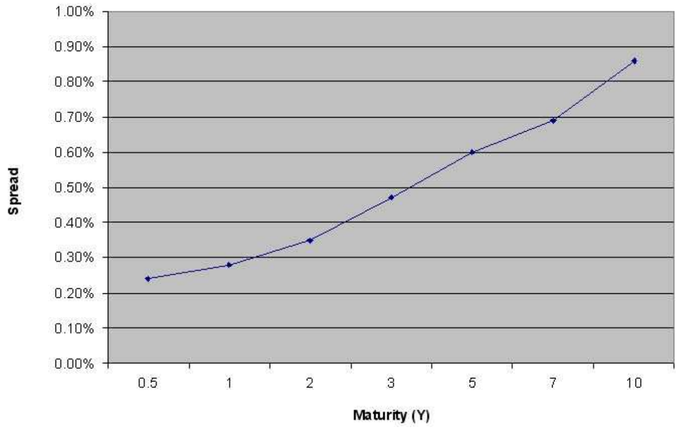

Figure 1: Spread curve for France-Telecom

Then, we bootstrap the market default probabilities from the quoted CDS spreads and we compare these probabilities with those obtained in ECMM:

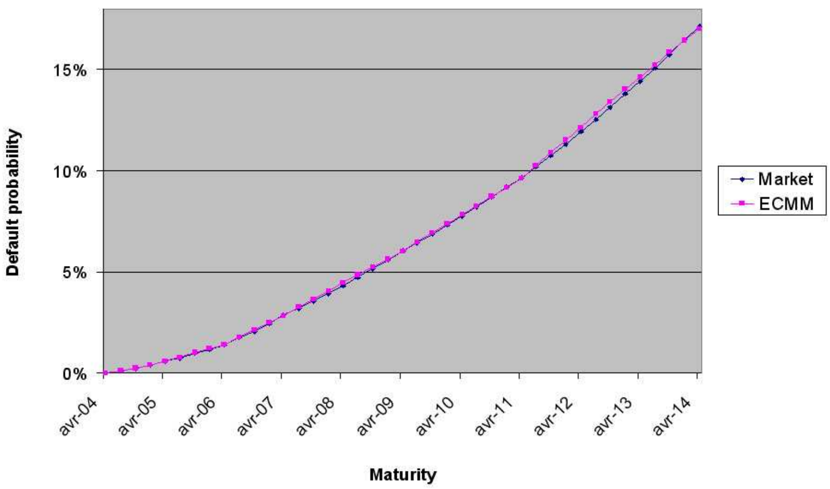

Figure 2: Comparison between ECMM default probabilities and market default probabilities

5. Pricing Credit Derivatives in an ECMM.

5.1. Default Payoffs.

We give in this section the present value of different default payoffs under the assumption of independence between credit and default-free interest rates.

Proposition 5.1. *Under the previous assumption, the present value of a payment of* 1 *at time* T_{k+1} *if a default occurs during the period* $]T_k, T_{k+1}]$ *is :*

$$e_k(0) = \delta_k \overline{B}_{k+1}(0) H_k(0)$$

Proof. The independence hypothesis gives immediately:

$$\mathbb{Q}[\tau > T_k] = D_k(0)$$

We thus have

$$\begin{aligned} e_k(0) &= \mathbb{E}^{\mathbb{Q}}\left[\beta(T_{k+1})\mathbf{1}_{\{T_k<\tau\leq T_{k+1}\}}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\beta(T_{k+1})\right] \times \mathbb{E}^{\mathbb{Q}}\left[(I(T_k) - I(T_{k+1}))\right] \\ &= B_{k+1}(0) \times (\mathbb{Q}[\tau > T_k] - \mathbb{Q}[\tau > T_{k+1}]) \\ &= B_{k+1}(0) \times (D_k(0) - D_{k+1}(0)) \\ &= \delta_k \overline{B}_{k+1}(0) H_k(0) \end{aligned}$$

■

5.2. Pricing methodology.

We now turn to the pricing of various credit derivatives within the ECMM. We make a clear distinction between two pricing methods. The first one only requires the simulation of the processes H_k and ε_k, which then appear in the Monte-Carlo formula. In the second one these processes are also simulated but only in order to get the default time information, and are no longer needed to compute the Monte-Carlo price of the product.

5.2.1. "Pricing Rule" Method : Application to Credit Spread Options. This method, as we shall see later, is particularly well-suited for mono-issuer products. It is mainly based on formula (12) from lemma 3.6. As an example, we consider the case of credit spread options (CSO) (other single-name products, such as CMDS or RMDS may be priced in a similar way).

Description. A European call with maturity T_K on a credit default swap (T_K, T_N) gives the buyer the right to enter at time T_K in a credit default swap over the period $]T_K, T_N]$ at a pre-determined (i.e. agreed at time 0) spread $s^*_{T_K,T_N}$ (strike).

Payoff. At a date t $(0 \leq t \leq T_K \leq T_N)$, the CDS with characteristics $(T_K, T_N, s^*_{T_K,T_N})$ has from the protection buyer point of view a value:

$$(1-R)\sum_{k=K}^{N-1} e_k(t) - s^*_{T_K,T_N}\sum_{k=K}^{N-1}\delta_k\overline{B}_{k+1}(t)$$

where $e_k(t)$ is the value at time t of a payment of 1 if a default occurs during the period $[T_k, T_{k+1}]$.

The CDS forward spread (or forward default swap rate), denoted as $s_{T_K,T_N}(t)$, is defined as the level of $s^*_{T_K,T_N}$ that makes the forward CDS fairly priced:

$$\begin{aligned} s_{T_K,T_N}(t) &= (1-R)\frac{\sum_{k=K}^{N-1} e_k(t)}{\sum_{k=K}^{N-1}\delta_k\overline{B}_{k+1}(t)} \\ &= (1-R)\frac{\sum_{k=K}^{N-1}\delta_k\overline{B}_{k+1}(t)H_k(t)}{\overline{\mathrm{BPV}}_{T_K,T_N}(t)} \end{aligned}$$

The time t - value of the forward CDS is thus:

$$\left[s_{T_K,T_N}(t) - s^*_{T_K,T_N}\right] \times \overline{\mathrm{BPV}}_{T_K,T_N}(t).$$

In case of survival until T_K, the option will be exercised only if its value at T_K is positive. Consequently, the payoff of a CSO can be written as:

$$I(T_K)[s_{T_K,T_N}(T_K) - s^*_{T_K,T_N}]_+\overline{\mathrm{BPV}}_{T_K,T_N}(T_K).$$

Pricing. The price at time $t = 0$ of a call with maturity T_K on an underlying CDS starting at T_K and ending at T_N and with strike $s^*_{K,N}$ can be expressed using the risk-neutral valuation formula:

$$C(0, T_K, T_N, s^*_{T_K,T_N}) = \mathbb{E}^{\mathbb{Q}}\left[\beta(T_K)I(T_K)\overline{\text{BPV}}_{T_K,T_N}(T_K) \times [s_{T_K,T_N}(T_K) - s^*_{T_K,T_N}]_+\right] \tag{24}$$

Using the DAP pricing rule, we can express this price as:

$$C(0, T_K, T_N, s^*_{K,N}) = \mathbb{E}^{\mathbb{Q}}\left[\beta(T_K)\varepsilon_K(T_K)\overline{\text{BPV}}_{T_K,T_N}(T_K) \times [s_{T_K,T_N}(T_K) - s^*_{T_K,T_N}]_+\right] \tag{25}$$

where we have, using relation (13) with $t = T_K$,

$$\overline{\text{BPV}}_{T_K,T_N}(T_K) = \frac{1}{\varepsilon_K(T_K)} \sum_{k=K}^{N-1} \delta_k \varepsilon_{k+1}(T_K) B_{k+1}(T_K).$$

In equation (24), we need to know the default time of the issuer to compute the price of the CSO (through the term $I(T_K)$). On the contrary, equation (25) allows for a direct (i.e. without simulating explicitly the default time) computation of the CSO price, which is more convenient from a computational point of view: in particular, we do not need to simulate uniform variables to estimate the default time of the issuer.

Numerical Results. In order to test the robustness of our pricing method, we compute prices of CSO for a given volatility parametrization of the processes H_k and we find the corresponding implicit Black volatility.

More precisely, we work within a one-dimensional Black-type model with a given volatility term structure. The model specification is a 3-month column. The purpose of the numerical test is to assert the accuracy of the ECMM. In particular, it seems important to verify that the use of the DAP does not introduce any simulation bias, so that the drift terms are well reproduced.

We have seen that:

$$\frac{ds_{T_K,T_N}(t)}{s_{T_K,T_N}(t)} = \sigma_{K,N} dW_t^{\overline{\mathbb{Q}}^{K,N}}$$

and the present value of the CSO is:

$$\begin{aligned} C(0, T_K, T_N, s^*_{T_K,T_N}, \sigma_{K,N}) &= \overline{\text{BPV}}_{T_K,T_N}(0) \times \mathbb{E}^{\overline{\mathbb{Q}}^{K,N}}\left[[s_{T_K,T_N}(T_K) - s^*_{T_K,T_N}]_+\right] \\ &= \overline{\text{BPV}}_{T_K,T_N}(0) \times \left[s_{T_K,T_N}(0)N(d_1) - s^*_{T_K,T_N}N(d_2)\right] \end{aligned}$$

where d_1 and d_2 are given by

$$d_{1,2} = \frac{\ln\left(\frac{s_{T_K,T_N}(0)}{s^*_{T_K,T_N}}\right) \pm \frac{1}{2}(\sigma^{K,N})^2 T_K}{\sigma_{K,N}\sqrt{T_K}}$$

The implied Black volatility of a CSO is defined by:

Definition 5.2. *Implied Black volatility*

Let $C^m(0, T_K, T_N, s^*_{T_K,T_N})$ be the market price of a Credit Spread Call with the same characteristics as above. The implied Black volatility $\sigma^*_{K,N}$ is defined by:

$$C(0, T_K, T_N, s^*_{T_K,T_N}, \sigma^*_{K,N}) = C^m(0, T_K, T_N, s^*_{T_K,T_N})$$

We then proceed as follows:

1. simulation (Monte Carlo) of the H_K and $\varepsilon_K(T_K)$ using arbitrary volatilities $\sigma_{K,K+1}$ for the forward spreads $s_{T_K,T_{K+1}}$.

2. computation of the CSO prices $C(0, T_K, T_{K+1}, s^*_{T_K,T_{K+1}}, \sigma^*_{K,K+1})$ where $\delta_K = 3$ months for all K (3-month column), for maturities varying from 3M to 10Y.
3. derivation of the implied volatilities $\sigma_{K,K+1}$ by inversion of the Black formula.

The following example shows that the ECMM reproduces very accurately the Black CSO prices even for high volatilities and long maturities:

Example 5.3. *The market data and model parameters are the same as in example 4.3.*

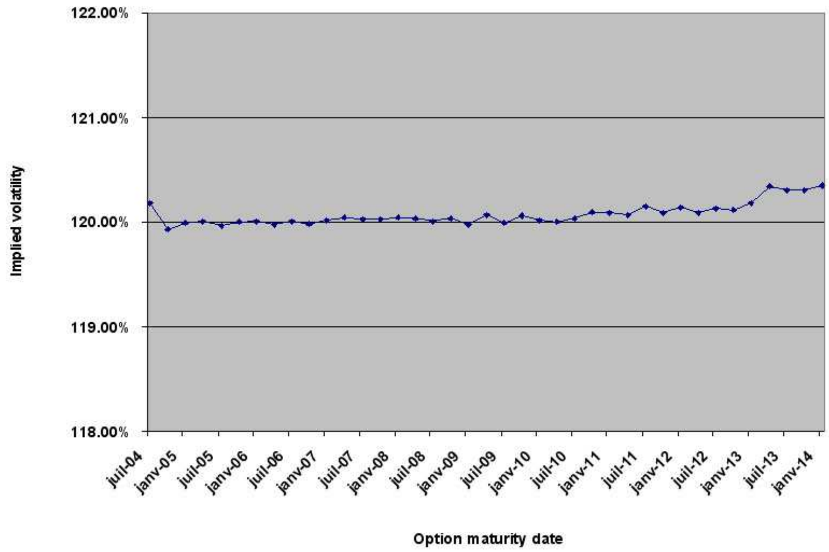

Figure 3: France-Telecom implied volatilities in a 1-column model

5.2.2. "Default Simulation" Method : Application to n^{th}-to-default and CDO. This second method is more general than the first one and should be used when the DAP pricing rule is no longer applicable, especially in the case of multi-issuer products.

We first describe the principle of this pricing method and we apply it to the pricing of n^{th}-to-default and CDO.

Principle. We assume that we want to estimate the price of a product whose payoff depends on the vector $(\tau_1, \tau_2, \ldots, \tau_n)$, when τ_i stands for the default time of the i^{th} issuer of a given basket (note that this method applies to mono-issuer products when $n = 1$). Consequently, the price of such products can be computed by a Monte-Carlo method when we proceed as follows for each trajectory:

1. simulation of the processes $H_k^{(i)}$ and $\varepsilon_k^{(i)}$ for all i (index for issuer i) and k (index of dates) under the risk-neutral measure, given some correlation matrices Ω_k between the $H_k^{(i)}$ and a volatility term-structure for each issuer;
2. determination of the default times τ_i of the issuers. If we still assume that an issuer can only default on a discrete set of dates $(T_k))_k$, then an approximation $\hat{\tau}_i$ of τ_i is given by formula (19):
$$\hat{\tau}_i = \inf\left\{T_k, \varepsilon_k^{(i)}(T_k) < U\right\}$$
where $U \sim \mathcal{U}([0,1])$ and is independent from the Brownian trajectory used for simulating the $H_k^{(i)}$.
3. computation of the payoff.

Remark 5.4. *We can simulate several default times realization for each issuer with only one trajectory of the DAP by using several independent variables $U_j \sim \mathcal{U}([0,1])$, which allows for a gain in computation time.*

Pricing of n^{th}-to-default. Assume we have a basket made of N issuers and we want to estimate the price of an n^{th}-to-default contract on this basket. We denote as $\tilde{\tau}$ the n^{th} default time among the N default times $\tau_1, \ldots, \tau_N$, I the index of the issuer which defaults at $\tilde{\tau}$ (we assume that two issuers cannot default at the same time) and R_I the recovery rate associated with this issuer.

The payoff of the protection leg of n^{th}-to-default with maturity T is given by:

$$(1 - R_I)\mathbf{1}_{\{\tilde{\tau} \leq T\}}$$

so that its value at time $t = 0$ is given as:

$$\mathbb{E}^{\mathbb{Q}}\left[\beta_{\tilde{\tau}}(1 - R_I)\mathbf{1}_{\{\tilde{\tau} \leq T\}}\right]$$

if we assume that the protection payment is made at default.

Consequently, the computation of this price by Monte-Carlo is quite simple. For each Monte-Carlo trajectory:

1. we simulate the default times τ_i for issuers i, $1 \leq i \leq N$;
2. we find the nth-to-default issuer and the corresponding default time and recovery;
3. we compute the payoff for this trajectory.

The computation of the premium leg is done in the same way since it depends only on the default time $\tilde{\tau}$ and on default-free interest rates.

Pricing of CDO. Similarly, the pricing of CDO is straightforward using this pricing methodology. We first recall briefly the main characteristics of a CDO.

We consider a basket of N issuers with associated default times τ_i and recovery rates R_i. We denote as N_i the nominal of the i^{th} issuer in the CDO, and the corresponding loss is given as $L_i = (1 - R_i)N_i$. The cumulated loss up to time T is then:

$$\Lambda(T) := \sum_{i=1}^{N} L_i \mathbf{1}_{\{\tau_i \leq T\}}$$

We still assume that the defaults can only occur on a discrete set of dates $(T_k)_{1 \leq k \leq n}$. For a single-tranch CDO with maturity T, spread s^* and strikes K_1 and K_2 ($K_1 < K_2$), the payoffs are the following:

- floating leg: at each time $\overline{T}_k := \frac{1}{2}(T_{k-1} + T_k)$, the protection seller makes a payment equals:

$$\text{CS}(\Lambda(T_k), K_1, K_2) - \text{CS}(\Lambda(T_{k-1}), K_1, K_2)$$

  where

$$\text{CS}(\Lambda(T_k), K_1, K_2) = [\Lambda(T_k) - K_1]_+ - [\Lambda(T_k) - K_2]_+$$

- fixed leg: at each time T_k, the protection buyer pays:

$$s^* \delta_{i-1} \left(K_2 - K_1 - \frac{1}{2}\left[\text{CS}(\Lambda(T_{k-1}), K_1, K_2) + \text{CS}(\Lambda(T_k), K_1, K_2)\right] \right)$$

The present value of each leg equals the risk-neutral expectation of the discounted payoff. Consequently, it suffices to know the values of the quantities:

$$\mathbb{E}^{\mathbb{Q}}\left[\text{CS}(\Lambda(T_k), K_1, K_2)\right],$$

which is easy once we know how to simulate the variables $\Lambda(T_k)$.

6. Conclusion.

With the introduction of a specific class of process, the Default Accumulator Process, we have achieved to present a standard credit market model. This framework is flexible enough to allow for the pricing of any type of credit-linked products. The focus made on simulation and pricing indicates that the ECMM represents a tractable framework.

As the risk-neutral modelling allows for an easy adaptation of the classical methods commonly in the case of Libor Market Models, the next step may be to define an appropriate diffusion process for the underlying forward spread. Another crucial aspect may be to setup the correlation structure required when pricing multi-name products. These issues are let for further studies.

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