Market Completeness in the Presence of Default Risk

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1. Introduction

The completeness for a market including default risk is classically achieved using the extension of a default-free market. The line of argument consists in enlarging the initial default-free and arbitrage-free market, which is also supposed to be complete, so that default risk can be introduced. Then, the completeness of the resulting market is proved for a given class of defaultable claims. Blanchet-Scalliet and Jeanblanc (2004) give a construction of this type, and show that the defaultable market is complete as soon as a defaultable zero-coupon bond is traded. Similar result can be found in Bielecki and Rutkowski (2001), and in Bélanger et al. (2001). The construction is achieved either by the use of a suitable representation theorem, as in [4], or by direct proof, as in [3]. The main difficulties lie in the rebate parts, also called payments at hit, and in the practical interpretation of the hedging portfolio. Particular focus on these aspects can be found in [4]; related results can also be found in [2].

We study the case of a credit market and try to achieve the completeness, for a given class of assets, using replicating strategy based on a set of defaultable basic assets and cash account. In practice, the assets used for hedging strategies are the defaultable zero-coupon bonds of different maturities. As the payment at hit is admissible in the credit market, this class of asset can be considered as equivalent to the CDS, which is the practical hedging instrument.

The general framework is a reduced form setup in which the information flow is modelised through a brownian filtration $\mathbb{F}$, which is augmented with the default time information. The market is supposed to be frictionless and arbitrage-free.

The main result is that, for an information flow obtained from a d-dimensional brownian motion, the hedging strategy involves exactly $d+1$ defaultable zero-coupon bonds of different maturities.

Under some technical conditions, the replicating portfolio can be built, being however difficult to interprete. Specific examples are then provided, so that replicating strategies associated to simple contracts can be analysed.

The paper is made as follows: in the first section, the basic notations and results for a standard reduced form model are briefly recalled. In the second section, the standard construction of a complete market including default risk is presented, in order to make explicit the usual tools and processes. The third and fourth sections develop the construction of completeness for a credit market. The first step is to achieve this for a defaultable zero-coupon bond market. Then, by including payment at hits, the result is extended to the general case of a CDS market. Examples of hedging strategies are provided for both cases.

2. Intensity-Based Default Model Setup

In this section, the standard intensity-based approach is presented briefly. Main notations and useful results are recalled, and will be used later in the different market constructions given below.

Similar presentation and connected results can be found in [6], [5] and [7].

2.1. Basic Results and Notations. The default time τ is supposed to be a non-negative random variable on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, satisfying $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for all $t \in \mathbb{R}_+$. We define the right-continuous process H as:

$$H_t = \mathbf{1}_{\{\tau \leq t\}}$$

and the associated filtration $\mathbb{H}$ as:

$$\mathcal{H}_t = \sigma(H_u : u \leq t)$$

We assume the existence of an auxiliary filtration $\mathbb{F}$ such that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, and for all $t \in \mathbb{R}_+$, $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. In practice, $\mathbb{F}$ often corresponds to the modelisation of the uncertainty relative to the default-free market, and is taken as a Brownian filtration, satisfying the usual conditions. Note that as $H_t \in \mathcal{G}_t$, τ is a $\mathcal{G}$-stopping time, but it is not an $\mathbb{F}$-stopping time.

The interest rate is supposed to be a non-negative process, and we denote $\beta_t = \exp\left(\int_0^t r_s ds\right)$ the savings account.

Hazard Process. We set $F_t = \mathbb{P}(\tau \leq t \,|\mathcal{F}_t)$, for all $t \in \mathbb{R}_+$. Since, for any $s \geq t$,

$$\mathbb{E}[F_s \,|\mathcal{F}_t] = \mathbb{E}[\mathbb{P}(\tau \leq s \,|\mathcal{F}_s) \,|\mathcal{F}_t] = \mathbb{P}(\tau \leq s \,|\mathcal{F}_t) \geq \mathbb{P}(\tau \leq t \,|\mathcal{F}_t)$$

F is a bounded non-negative sub-martingale. Then, F can be be taken as RCLL.

Definition 1 [$\mathbb{F}$-Hazard Process]**.** *If $F_t < 1$ for all $t \in \mathbb{R}_+$, then the $\mathbb{F}$-Hazard Process of τ is defined as $\Gamma_t = -\ln(1 - F_t)$. Equivalently, $1 - F_t = \exp(-\Gamma_t)$.*

It is clear from the definition of F that $\Gamma_0 = 0$.

The following classical result will be used intensively in what follows:

Lemma 2. *Let Y be a $\mathcal{G}$-measurable random variable and let $t \leq s$. Then*

$$\mathbb{E}\left[\mathbf{1}_{\{\tau > s\}} Y \,|\mathcal{G}_t\right] = \mathbf{1}_{\{\tau > t\}} \mathbb{E}\left[\mathbf{1}_{\{\tau > s\}} \exp(\Gamma_t) \, Y \,|\mathcal{F}_t\right]$$

and

$$\mathbb{E}\left[\mathbf{1}_{\{t < \tau \leq s\}} Y \,|\mathcal{G}_t\right] = \mathbf{1}_{\{\tau > t\}} \mathbb{E}\left[\mathbf{1}_{\{t < \tau \leq s\}} \exp(\Gamma_t) \, Y \,|\mathcal{F}_t\right]$$

If Y is $\mathcal{F}_s$-measurable, then

$$\mathbb{E}\left[\mathbf{1}_{\{\tau > s\}} Y \,|\mathcal{G}_t\right] = \mathbf{1}_{\{\tau > t\}} \mathbb{E}\left[\exp(\Gamma_t - \Gamma_s) \, Y \,|\mathcal{F}_t\right] \tag{1}$$

A demonstration can be found in [3].

Martingale Associated with the Hazard Process. We define the process L as:

$$L_t = \mathbf{1}_{\{\tau > t\}} \exp(\Gamma_t) = (1 - H_t) \exp(\Gamma_t) \tag{2}$$

Lemma 3. *The process L is a $\mathbb{G}$-martingale.*

Lemma 4. *If Γ is a continuous increasing process, then the process $M_t = H_t - \Gamma_{t \wedge \tau}$ is a $\mathbb{G}$-martingale, and the process L satisfies:*

$$L_t = 1 - \int_{]0,t]} L_{u-} dM_u$$

Proof. From (2), and using $\Gamma_0 = 0$, it comes directly that $L_0 = 1$. Applying integration by part formula to (2):

$$L_t = 1 + \int_{]0,t]} \exp(\Gamma_u) \left[(1 - H_u) d\Gamma_u - dH_u\right] \tag{3a}$$

As $\Gamma_{t\wedge\tau} = \int_0^{t\wedge\tau} d\Gamma_u = \int_{]0,t]} (1 - H_u) d\Gamma_u$, M_t can be written as:

$$M_t = \int_{]0,t]} (dH_u - (1 - H_u) d\Gamma_u)$$

so that:

$$M_t = \int_{]0,t]} \exp(-\Gamma_u) dL_u$$

■

Self-Financing Strategy. We recall the definition of a self-financing strategy (see [8]):

Definition 5. *A self-financing strategy is defined as a couple of adapted processes $\left(\eta_t^0\right)_{0\le t\le T^*}$ and $(\eta_t)_{0\le t\le T^*}$ such that:*

- $\int_0^T \left|\eta_t^0\right| dt + \int_0^T \|\eta_t\|^2 dt < +\infty$
- $\eta_t^0 \beta_t + \eta_t \cdot S_t = \eta_0^0 \beta_0 + \eta_0 S_0 + \int_0^t \eta_u^0 dS_u^0 + \int_0^t \eta_u \cdot dS_u$ *a.s.* $\forall t \in [0, T^*]$

where β_t is the cash account and S the risky assets for a given market.

Typically, S represent risky assets in the default-free market, and are $\mathcal{F}$-adapted.

2.2. Hypothesis.

General Hypothesis. The market of default-free and defaultable claims presents the following properties:

- it is frictionless,
- it allows continuous trading over a finite period of time $[0, T^*]$, for a given maturity $T^* > 0$.

Specific Hypothesis.

- the uncertainty in the default-free market is modelised through the reference filtration $\mathbb{F}$, for the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
- the default-free market is complete and arbitrage free. In particular there exists a unique martingale measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, on $(\Omega, \mathcal{F}_{T^*})$.

Technical Conditions.

- the martingale invariance property holds under the equivalent martingale measures, i.e.: any square integrable $\mathbb{F}$-martingale under $\mathbb{P}^*$ follows a $\mathbb{G}$- martingale under $\mathbb{P}^*$,
- the $\mathbb{F}$-martingales are continuous,
- the $\mathbb{F}$-hazard process Γ of τ is continuous.

3. Market Completeness: Extension of the Default-Free Market

In this section, we achieve market completeness for a market including default risk by extending an initial default-free complete market. This is the usual approach, and it allows to introduce the main arguments and needed technical tools. This will also allow for comparison, in terms of hedging strategy, with the other construction presented hereafter.

We assume that at least one defaultable zero-coupon of maturity $T < T^*$ is traded in the market, and make the assumption of zero recovery in case of default. Then, we examine the replication of defaultable claims with zero recovery. In particular, the issue of payments at hit is not looked at in this part.

As the initial default-free market is complete, we have the following lemma:

Lemma 6. *$\forall X$ $\mathcal{F}_{T^*}$-measurable, $\mathbb{P}^*$-integrable random variable, X admits a self-financing replicating strategy.*

This is simply the consequence of the hypothesis of completeness for the risk-free market.

We then assume that the defaultable zero-coupon bond has zero recovery, so that its price is given by:

$$\overline{P}(t,T) = \beta_t \mathbb{E}^{\mathbb{Q}^*}\left[\beta_T^{-1}\mathbf{1}_{\{\tau>T\}}\,|\mathcal{G}_t\right] \tag{4}$$

where $\mathbb{Q}^*$ is a probability measure, equivalent to $\mathbb{P}$, chosen by the market.

In fact, the market gives the price of the defaultable zero-coupon bond such that it does not induce arbitrage opportunity. When defaultable zero-coupon of all maturities $T < T^*$ are traded, it is easy to check that there exists a unique probability measure such that (4) holds and such that the restriction of $\mathbb{Q}^*$ to $\mathcal{F}$ is equal to $\mathbb{P}^*$.

3.1. Dynamic of the defaultable zero-coupon bond price. We start by setting:

$$m_t = \mathbb{E}^{\mathbb{Q}^*}\left[\beta_T^{-1}\exp\left(-\Gamma_T\right)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}^*}\left[\beta_T^{-1}\exp\left(-\Gamma_T\right)|\mathcal{F}_t\right]$$

where this equality follows from the fact that $\mathbb{Q}^*$ and $\mathbb{P}^*$ coincides on $\mathcal{F}_{T^*}$. The process m_t follows an $\mathbb{F}$-martingale on $\mathbb{P}^*$, and on $\mathbb{Q}^*$.

Corollary 7. *If the price of the defaultable zero-coupon bond is given by* (4)*, then:*

$$d\overline{P}\left(t,T\right) = \overline{P}\left(t,T\right)\left(r_t dt - dM_t\right) + \beta_t L_{t-} dm_t$$

Proof. *Starting from* (4)*, and using the fundamental equality* (1)*, we have* $\overline{P}\left(t,T\right) = L_t\beta_t m_t$*. Applying Itô lemma then leads to:*

$$\begin{aligned} d\overline{P}\left(t,T\right) &= \beta_t L_{t-} dm_t + \beta_t m_t dL_t + L_t m_t d\beta_t \\ &= r_t\overline{P}\left(t,T\right)dt + \beta_t L_{t-} dm_t + \beta_t m_t dL_t \end{aligned}$$

As $dL_t = -L_{t-}dM_t$*,*

$$d\overline{P}\left(t,T\right) = r_t\overline{P}\left(t,T\right)dt + \beta_t L_{t-} dm_t - \beta_t m_t L_{t-} dM_t$$

■

We recall that the defaultable zero-coupon bond with zero recovery is included in the set of tradable assets available in the market.

We also set:

$$Z\left(t,T\right) = \overline{P}\left(t,T\right)\beta_t^{-1}$$

the discounted price process of the defaultable zero-coupon bond.

3.2. Replicating Strategy. To show that the defaultable market is complete, we proceed as follows:

- we postulate that the price of any defaultable security is given through the usual risk-neutral expectation formula,
- we construct a self-financing replicating strategy, consisting in continuous trading of default-free securities and in defaultable zero-coupon bonds,
- we conclude that the price is effectively given by the risk-neutral formula, and that the market is complete.

We consider a general defaultable claim $(X, 0, \tau)$, that pays X at maturity T in case of no default, and zero otherwise. The random variable X is supposed to be $\mathcal{F}_T$-measurable and integrable with respect to $\mathbb{Q}^*$. We set S_t^0 the price process of the defaultable claim and $\widetilde{S}_t^0 = S_t^0\beta_t^{-1}$ its discounted price process. As said previously, we postulate that:

$$\widetilde{S}_t^0 = \mathbb{E}^{\mathbb{Q}^*}\left[\beta_T^{-1}X\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t\right]$$

We know that:

$$\widetilde{S}_t^0 = \mathbf{1}_{\{\tau>t\}}\exp\left(\Gamma_t\right)\mathbb{E}^{\mathbb{Q}^*}\left[\beta_T^{-1}X\exp\left(-\Gamma_T\right)|\mathcal{F}_t\right] = L_t m_t^X$$

where

$$m_t^X = \mathbb{E}^{\mathbb{Q}^*}\left[\beta_T^{-1} X \exp\left(-\Gamma_T\right) | \mathcal{F}_t\right]$$

and $\left(m_t^X\right)_t$ is an $\mathbb{F}$-martingale under $\mathbb{Q}^*$ (and also under $\mathbb{P}^*$).

This leads to the following representation theorem for $\widetilde{S}^0$:

Lemma 8. *The G-martingale $\widetilde{S}^0$ admits the integral representation*

$$\widetilde{S}_t^0 = \widetilde{S}_0^0 + \int_0^{t\wedge\tau} \exp\left(\Gamma_u\right) dm_u^X - \int_{]0,t\wedge\tau]} \exp\left(\Gamma_u\right) m_u^X dM_u$$

Proof. *Starting from $\widetilde{S}_t^0 = L_t m_t^X$, and applying the Itô product rule with the fact that m_t^X is continuous leads to:*

$$\widetilde{S}_t^0 = \widetilde{S}_0^0 + \int_0^t L_{u-} dm_u^X - \int_{]0,t]} \exp\left(\Gamma_u\right) m_u^X dM_u$$

We conclude with the property that M is stopped at τ and $L_t = \mathbf{1}_{\{\tau>t\}} \exp\left(\Gamma_t\right)$. ■

Recall that the hypothesis of completeness for the default-free market implies that $Y_1 = \exp\left(-\Gamma_T\right)$ and $Y_2 = X\exp\left(-\Gamma_T\right)$ admit self-financing replicating strategy involving only default-free securities. Without loss of generality, we then consider these two assets as primary securities in what follows. It is interesting to note that the discounted price processes of Y_1 and Y_2 are respectively given by m_t and m_t^X.

Proposition 9. *Let us denote $\zeta_t^X = m_t^X m_t^{-1}$. On the set $\{\tau \geq t\}$, the replicating strategy for the discounted price $\widetilde{S}_t^0$ is given by the portfolio:*

$$\begin{aligned} \phi_t^0 &= \zeta_t^X \\ \phi_t^1 &= -\exp\left(\Gamma_t\right)\zeta_t^X \\ \phi_t^2 &= \exp\left(\Gamma_t\right) \end{aligned}$$

where the corresponding hedging instruments are: the discounted price process $Z\left(t,T\right)$ of the defaultable zero-coupon bond, and the discounted process of default-free claims $Y_1 = \exp\left(-\Gamma_T\right)$ and $Y_2 = X\exp\left(-\Gamma_T\right)$.

In particular, the replicating strategy verifies $\phi_t^1 m_t + \phi_t^2 m_t^X = 0$.

On the set $\{\tau < t\}$, the strategy is identically equal to zero.

Proof. As the discounted price processes of Y_1 and Y_2 are respectively given by m_t and m_t^X. We have:

$$\begin{aligned} dZ\left(t,T\right) - L_{t-} dm_t &= m_t dL_t \\ &= -L_{t-} m_t dM_t \\ &= -\exp\left(-\Gamma_t\right) m_t dM_t \end{aligned}$$

Combining this equation with the martingale representation theorem:

$$\begin{aligned} \widetilde{S}_t^0 &= \widetilde{S}_0^0 + \int_0^{t\wedge\tau} \exp\left(\Gamma_u\right) dm_u^X - \int_{]0,t\wedge\tau]} \exp\left(\Gamma_u\right) m_u^X dM_u \qquad (5) \\ &= \widetilde{S}_0^0 + \int_{]0,t\wedge\tau]} \zeta_u^X dZ\left(u,T\right) - \int_{]0,t\wedge\tau]} \exp\left(\Gamma_u\right) \zeta_u^X dm_u + \int_0^{t\wedge\tau} \exp\left(\Gamma_u\right) dm_u^X \end{aligned}$$

which completes the proof. ■

As $\widetilde{S}_t^0 = \zeta_t^X Z(t,T) - \exp(\Gamma_t)\,\zeta_t^X m_t + \exp(\Gamma_t)\, m_t^X$, then (5) means that the chosen strategy is self-financing.

4. DEFAULTABLE ZERO-COUPON BOND MARKET

We now turn to a different construction, and try to show that the market of defaultable zero-coupon bonds is complete.

Two important additional hypotheses are then needed. The first one is technical, as it specifies that the information flow is modelised through a d-dimensional Brownian motion. The second one states that a continuum of defaultable zero-coupon bonds is traded in the market.

The construction shares a lot of the technical tools with the preceding section, and the line of arguments is also similar, in that it consists in building the hedging strategy, for a given class of assets, using the liquid market products. However, the process and the result are different in that the replicating strategy involves only defaultable claims and cash account.

The completeness of the defaultable zero-coupon market is the first step in achieving to show the completeness of the credit default swap market. This is due to the class of admissible assets, and more specifically to the fact that the defaultable zero-coupon market does not include payments at hit. As this represents the only major difference between the two markets, it is let to a latter examination.

We start by defining a defaultable zero-coupon bond market, and the associated admissible claims. Then, the replication strategy, based on a specified set of defaultable zero-coupon, is build, so that the completeness is achieved under some technical conditions. The section ends with an example of replicating strategy for which a financial interpretation of the portfolio coefficients is made possible.

4.1. Model Set-Up. We keep the same notations as in the preceding section, and recall the following useful result (see [8]):

Theorem 10. *Let $W = \left\{W_t = \left(W_t^1, ..., W_t^d\right); 0 \le t < +\infty\right\}$ be a d-dimensional Brownian motion on $(\Omega, \mathcal{F}_t, \mathbb{P})$, and let $\{\mathcal{F}_t\}$ be the augmentation under $\mathbb{P}$ of the filtration $\left\{\mathcal{F}_t^W\right\}$ generated by W. Then, for any square-integrable martingale $M = \{M_t, \mathcal{F}_t; 0 \le t < +\infty\}$ with $M_0 = 0$ a.s., there exists $\mathcal{F}_t$ predictable processes $Y^{(j)} = \left\{Y_t^{(j)}, \mathcal{F}_t; 0 \le t < +\infty\right\}$ such that:*

$$\mathbb{E}\int_0^T \left(Y_t^{(j)}\right)^2 dt < +\infty$$

for every $0 < T < +\infty$, and

$$M_t = \sum_{j=1}^d \int_0^t Y_s^{(j)} dW_s^{(j)}$$

for every $0 \le t < +\infty$

We assume that the filtration $\mathcal{F}_t$, representing the uncertainty relative to the default-free market, is a Brownian filtration generated by $W = \left\{W_t = \left(W_t^1, ..., W_t^d\right), \mathcal{F}_t; 0 \le t < +\infty\right\}$.

We then introduce a set of $d+1$ maturities $(T_k)_{k=0,..,d}$ and corresponding defaultable zero-coupon bonds $\left(\overline{P}(t,T_k)\right)_{k=0,...,d}$. The market is then constituted by the $d+1$ defaultable zero-coupon bonds and the savings account.

The set of admissible claims is defined as follows:

Definition 11. *The claims that are allowed in the market are of the form $X\mathbf{1}_{\{\tau>T\}}$, with X being $\mathcal{F}_T$-measurable and $\mathbb{P}^*$ integrable.*

It represents the class of assets for which a replication strategy, in terms of defaultable zero-coupon plus savings account, is to be built.

4.2. Dynamic of the defaultable zero-coupon bond price.

We start by defining the following processes, for $\forall i \in \{0,...,d\}$:

$$m_t^i = \mathbb{E}^{\mathbb{Q}^*}\left[\beta_{T_i}^{-1}\exp\left(-\Gamma_{T_i}\right)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}^*}\left[\beta_{T_i}^{-1}\exp\left(-\Gamma_{T_i}\right)|\mathcal{F}_t\right]$$

Using the representation theorem of Brownian, square integrable martingales, we have the existence of processes $\left(\phi_t^0,...,\phi_t^d\right)_{0\le t\le T^*}$ such that:

- $\mathbb{E}^{\mathbb{Q}^*}\int_0^T\left(\left(\phi_t^{i,j}\right)\right)^2 dt < +\infty$ for every i,j in $\{0,...,d\}$
- $dm_t^i = \phi_t^i dW_t$ for every $i \in \{0,...,d\}$

with each of the introduced processes being d-dimensional.

Corollary 12. *If the price of the defaultable zero-coupon bond is given by* (4)*, then for all $i \in \{0,...,d\}$:*

$$d\overline{P}(t,T_i) = \overline{P}(t,T_i)\left(r_t dt - dM_t\right) + \beta_t L_{t-} dm_t^i$$

We then set $\left(Z^i(t,T_i)\right)_{i=0,...,d}$ the price processes of discounted defaultable zero-coupon bonds:

$$\begin{aligned} dZ(t,T_i) &= L_{t-}dm_t^i - L_{t-}m_t^i dM_t \\ &= L_{t-}dm_t^i - \exp\left(\Gamma_t\right)m_t^i dM_t \end{aligned} \tag{6}$$

4.3. Replicating Strategy.

To show that the defaultable market is complete, we proceed as follows:

- we postulate that the price of any defaultable security is given through the usual risk-neutral expectation formula,
- we construct a self-financing replicating strategy, consisting in continuous trading in defaultable zero-coupon bonds,
- we conclude that the price is effectively given by the risk-neutral formula, and that the market is complete.

Construction of a set of basic processes. For the sake of simplicity, we introduce a set of d basic assets $\left(A_t^i\right)_{i=1,..,d}$ defined as follows:

$$dA_t^i = \exp\left(\Gamma_t\right) m_t^i dZ\left(t, T_0\right) - \exp\left(\Gamma_t\right) m_t dZ\left(t, T_i\right) \tag{7}$$

for every $i \in \{1, ..., d\}$. Then:

$$dA_t^i = L_{t-} \exp\left(\Gamma_t\right) \left(m_t^i dm_t^0 - m_t^0 dm_t^i\right)$$

The martingale representation theorem implies that:

$$dA_t^i = L_{t-} \exp\left(\Gamma_t\right) \left(m_t^i \phi_t - m_t^0 \phi_t^i\right) dW_t$$

as defined in the previous section. By setting:

$$\psi_t^i = \exp\left(\Gamma_t\right) \left(m_t^i \phi_t - m_t \phi_t^i\right)$$

we get

$$dA_t^i = L_{t-} \psi_t^i dW_t$$

Re-writting the preceding equation under a matricial form leads to:

$$dA_t = L_{t-} \Sigma dW_t$$

where Σ is defined by $\Sigma_{i,j} = \psi_{i,j}\left(t\right)$.

Main Hypothesis. The main hypothesis that is necessary for the construction of the replicating strategy concerns the matrix Σ.

Condition 13. *The matrix Σ is invertible.*

This condition is required only on the set $\{\tau > t\}$ since the replicating strategy is identically zero on the set $\{\tau \leq t\}$, as we have assumed zero recovery in case of default. Note that this condition is satisfied in the case $d = 1$.

As in the previous section, we define:

$$\widetilde{S}_t^0 = \mathbb{E}^{\mathbb{Q}^*}\left[\beta_T^{-1} X \mathbf{1}_{\{\tau > T\}} \left|\mathcal{G}_t\right.\right]$$

We know that:

$$\widetilde{S}_t^0 = \mathbf{1}_{\{\tau > t\}} \exp\left(\Gamma_t\right) \mathbb{E}^{\mathbb{Q}^*}\left[\beta_T^{-1} X \exp\left(\Gamma_T\right) \left|\mathcal{F}_t\right.\right] = L_t m_t^X$$

where

$$m_t^X = \mathbb{E}^{\mathbb{Q}^*}\left[\beta_T^{-1} X \exp\left(\Gamma_T\right) \left|\mathcal{F}_t\right.\right]$$

and $\left(m_t^X\right)_{0 \leq t \leq T^*}$ is a $\mathbb{F}$-martingale under $\mathbb{Q}^*$ (and also under $\mathbb{P}^*$). Again, we use the martingale representation theorem for Brownian, square integrable martingale, so that there exists a process $\left(\phi_t^X\right)_{0 \leq t \leq T^*}$ verifying:

- $\mathbb{E}^{\mathbb{Q}^*} \int_0^{T^*} \left(\left(\phi_t^X\right)_j\right)^2 dt < +\infty$ for every $j \in \{1, ..., d\}$

- $dm_t^X = \phi_t^X dW_t$

The martingale representation theorem on $\widetilde{S}_t^0$ leads to:

Lemma 14. *The G-martingale $\widetilde{S}^0$ admits the integral representation*

$$\widetilde{S}_t^0 = \widetilde{S}_0^0 + \int_0^{t\wedge\tau} \exp(\Gamma_u)\, dm_u^X - \int_{]0,t\wedge\tau]} \exp(\Gamma_u)\, m_u^X dM_u$$

Self-Financing Replicating Strategy. Using (5):

$$\widetilde{S}_t^0 = \widetilde{S}_0^0 + \int_{]0,t\wedge\tau]} \zeta_u^X dZ(u,T_0) - \int_{]0,t\wedge\tau]} \exp(\Gamma_u)\, \zeta_u^X dm_u + \int_0^{t\wedge\tau} \exp(\Gamma_u)\, dm_u^X$$

The decomposition of $(m_t)_{0\le t\le T^*}$ and $\left(m_t^X\right)_{0\le t\le T^*}$ leads to:

$$\widetilde{S}_t^0 = \widetilde{S}_0^0 + \int_{]0,t\wedge\tau]} \zeta_u^X dZ(u,T_0) - \int_{]0,t]} L_{u-}\left(\zeta_u^X \phi_u^X + \phi_u\right) dW_u$$

As Σ is invertible, we have:

$$L_{t-}dW_t = \Sigma^{-1} dA_t$$

which means that:

$$\widetilde{S}_t^0 = \widetilde{S}_0^0 + \int_{]0,t\wedge\tau]} \zeta_u^X dZ(u,T) - \int_{]0,t]} \left(\zeta_u^X \phi_u^X + \phi_u\right) \Sigma^{-1} dA_u$$

and, as A_t is zero for $\tau \ge t$:

$$\widetilde{S}_t^0 = \widetilde{S}_0^0 + \int_{]0,t\wedge\tau]} \zeta_u^X dZ(u,T_0) - \int_{]0,t\wedge\tau]} \left(\zeta_u^X \phi_u^X + \phi_u\right) \Sigma^{-1} dA_u \tag{8}$$

This equation means that $\widetilde{S}_t^0$ is the price of a replicating strategy of the contingent claim paying $X\mathbf{1}_{\{\tau>T\}}$ at maturity T. The strategy involved is self-financing as the portfolio coefficients satisfy the integrability condition.

As the portfolio decomposition is invariant over any change of equivalent measure (this is due to the martingale representation theorem), we may conclude that the martingale measure $\mathbb{Q}^*$ is unique and that the price of any admissible contingent claim is given through the usual risk-neutral valuation formula.

4.4. Example: Hedging of CDS. In the previous section, the completeness of the defaultable zero-coupon bond market was proved. Yet, the hedging strategy proposed is difficult to interpret in terms of concrete contracts available currently in the credit market..

In order to illustrate the construction of the hedging strategy, we take the practical example of a CDS contract, for which we make explicit the hedging strategy, in terms of defaultable bonds, and give a financial interpretation for the portfolio coefficients.

In the particular case of a defaultable bond market, we define the Credit Default Swap contract in a slightly different way as the market do classically. The CDS is still a contract in which the protection buyer pays a periodical premium π up to default, and receives

$1-R$, where R stands for the recovery, in case of default. The difference consists in the payments timing. As the attainable claims are defined to be of the form $X\mathbf{1}_{\{\tau>T\}}$ for some $\mathcal{F}_t$-measurable random variable X, it is not possible to define payment occurring exactly at default, i.e. at τ. We thus give two different definitions of the CDS contract.

Definition 15 [Market CDS Contract]**.** *A Market CDS is defined as follows:*

- *the protection buyer receives* $1-R$ *in case of default, at time* τ*, and nothing otherwise*
- *the protection buyer pays a periodical premium* π *up to default, with an accrued coupon for the truncated period for which the default has happened*

Definition 16 [Discrete Tenor CDS Contract]**.** *A Discrete Tenor CDS is defined using a schedule* $(T_k)_{k\in[0..N]}$ *for which:*

- *the protection buyer receives* $1-R$ *in case of default, paid at time* $T_{\kappa(t)}$*, with* $\kappa(t)=\inf\{k/T_k>\tau\}$*, and nothing otherwise*
- *the protection buyer pays a periodical premium* π *up to default, with a plain coupon paid on the period for which the default has happened*

The CDS defined in this section will always be discrete tenor CDS contract. This definition is easily extendable to forward contracts, as soon as the forward period $[T_K,T_N]$ is given. Thus, we take as given a schedule $(T_k)_{k\in[0..N]}$, and define

$$e_k(t)=\mathbb{E}^{\mathbb{Q}^*}\left[\beta_{T_{k+1}}^{-1}\mathbf{1}_{\{T_k<\tau\le T_{k+1}\}}\,|\mathcal{G}_t\right]$$

We make the additional hypothesis of independence between interest rates and credit, which allows to make explicit calculation:

Definition 17 [Independence between credit and interest rates]**.** *The independence between credit and interest rates is defined for* $\mathbb{F}$*-measurable processes. Let* $(X_t,t\ge 0)$ *represents a pure credit process and* $(Y_t,t\ge 0)$ *a interest rate process, then* X_t *and* Y_t *are independent.*

Remark 18. *A simple way to achieve independence between credit and interest rates in the case of* $\mathcal{F}_t$ *being a Brownian filtration, is to define credit-linked processes relatively to the first* K *Brownian motions* $\left(W_t^k\right)_{t\ge0}^{k=1,..,K}$ *and to define other, non credit-linked processes with regards to the other Brownian motions* $\left(W_t^k\right)_{t\ge0}^{k=K+1,..,d}$.

Then, e_k becomes:

$$\begin{aligned} e_k(t) &= \beta_t\mathbb{E}^{\mathbb{Q}^*}\left[\beta_{T_{k+1}}^{-1}\mathbf{1}_{\{\tau>T_k\}}\,|\mathcal{G}_t\right]-\beta_t\mathbb{E}^{\mathbb{Q}^*}\left[\beta_{T_{k+1}}^{-1}\mathbf{1}_{\{\tau>T_{k+1}\}}\,|\mathcal{G}_t\right] \\ &= \frac{P(t,T_{k+1})}{P(t,T_k)}\overline{P}(t,T_k)-\overline{P}(t,T_{k+1}) \end{aligned}$$

for $t < T_k$. The definition of the discrete tenor forward CDS contract for period $[T_K, T_N]$ implies that the premium payments are directlty replicable in terms of defaultable zero-coupons. Thus, only the redemption payment is examined in what follows. Using the processes $e_k(t)$, we have the expression of the redemption payment as:

$$\begin{aligned} RP_{K,N}(t) &= \sum_{k=K}^{N-1} e_k(t) \\ &= \sum_{k=K}^{N-1} \left\{ \frac{P(t,T_{k+1})}{P(t,T_k)} \overline{P}(t,T_k) - \overline{P}(t,T_{k+1}) \right\} \end{aligned} \tag{9}$$

for all $t < T_k$ and $t < \tau$. This may be rewritten as:

$$RP(t) = \sum_{k=K}^{N-1} \theta_k(t) \overline{P}(t,T_k) \tag{10}$$

with

$$\theta_k(t) = \begin{cases} \frac{B_{K+1}(t)}{B_K(t)} \text{ if } k = K \\ \frac{B_{k+1}(t)}{B_k(t)} - 1 \text{ if } K < k < N \\ -1 \text{ if } k = N \end{cases}$$

The coefficients θ_k clearly defines a replicating strategy, consisting of defaultable zero-coupon bonds and cash account. This strategy can be given in a form very close to (8) by differentiating (10).

However, expression (9) appears to be more interesting. Indeed, it shows that a forward CDS redemption payment, for period $[T_K, T_N]$, is simply the sum of the forward CDS redemption payments for sub-periods $[T_k, T_{k+1}]$. Focusing then on the single period $[T_k, T_{k+1}]$, it comes:

$$RP_{k,k+1}(t) = \frac{P(t,T_{k+1})}{P(t,T_k)} \overline{P}(t,T_k) - \overline{P}(t,T_{k+1}) \tag{11}$$

Thus, the hedging portfolio consists in the defaultable zero-coupon of both maturities T_k and T_{k+1}, with very specific coefficients. In fact, the trading strategy associated consists in selling the defaultable zero-coupon bond for the highest maturity, and in buying the shortest in a quantity that corresponds exactly to the price at time t of the forward default-free zero-coupon bond. The composition of the hedging portfolio may be surprising at first, but it becomes quite natural when considering the realization of the strategy.

We then consider a short position in a forward CDS redemption payment, for the period $[T_k, T_{k+1}]$. Starting at t, the hedging portfolio is given by $\left(\frac{P(t,T_{k+1})}{P(t,T_k)}, -1\right)$. The portfolio is then dynamically modified so as to respect the hedging quantities at each date $t \leq s \leq T_k$. Depending on the arrival of default, two different scenarii are possible:

- If $\tau < T_k$: the forward CDS redemption payment has value zero, and so has the hedging portfolio.
- If $\tau \geq T_k$: the position in the T_k-defaultable zero-coupon has came to maturity, paying $P(T_k, T_{k+1})$. The resulting position at T_k is as follows: short position in CDS redemption payment and short position in the T_{k+1}-defaultable zero-coupon,

i.e. a position that will pay 1 at T_{k+1}, whatever default scenario may occur. This corresponds to a short position in the default-free zero-coupon, that can be unwinded using the cash account at T_k, since it has a value precisely of $P(T_k, T_{k+1})$.

5. Completeness of a CDS Market

Having achieved the completeness of a defaultable zero-coupon bond market, we now turn to the credit default swap market. As said previously, the CDS market can be seen as a defaultable zero-coupon bond market containing payments at hit. This is due to the decomposition of the CDS price in terms of defaultable zero-coupon and payment in case of default.

In this section, we achieve to include the payment at hit in the defaultable zero-coupon bond market, so that the standard CDS contract becomes part of the admissible claims.

5.1. Hedging a Payment at Hit. In this section, additional claims are added in the market, for which a replication strategy is needed:

Definition 19. *The claims that are added in the market are of the form $X_\tau \mathbf{1}_{\{\tau \le T\}}$, with X a bounded, $\mathcal{F}$-predictable process.*

In order to include this type of assets in the set of admissible claims, we present a construction of a replicating strategy, inspired by [4]. The replicating strategy will consists in a portfolio of defaultable zero-coupon and savings account.

To achieve this, we recall a result taken from [4].

Proposition 20. *Assuming the martingale invariance property holds, and the $\mathbb{F}$-hazard process of τ is continuous, let $M_t = H_t - \Gamma_{t\wedge\tau}$, and u be an $\mathbb{F}$-predictable process such that u_τ is integrable, and $H_t = \mathbb{E}\left[u_\tau \middle| \mathcal{G}_t\right]$. Then,*

$$U_t = m_0^u + \int_0^{t\wedge\tau} \exp\left(\Gamma_s\right) dm_s^u + \int_{]0,t\wedge\tau]} \left(u_s - U_{s-}\right) dM_s$$

where m^h is the $\mathbb{F}$-martingale

$$m_t^u = \mathbb{E}\left[\int_0^{+\infty} u_s dF_s \middle| \mathcal{F}_t\right]$$

We then introduce the discounted price of the payment at hit as:

$$\widetilde{P}_t = \mathbb{E}^{\mathbb{Q}^*}\left[\beta_\tau^{-1} X_\tau \mathbf{1}_{\{\tau\le T\}} \left|\mathcal{G}_t\right.\right]$$

Apllying the previous result to the process $u_t = \beta_t^{-1} X_t \mathbf{1}_{\{t\le T\}}$ leads to:

$$\widetilde{P}_t = m_0^X + \int_0^{t\wedge\tau} \exp\left(\Gamma_s\right) dm_s^X + \int_{]0,t\wedge\tau]} \left(X_s - \widetilde{P}_{s-}\right) dM_s$$

where m^u is the $\mathbb{F}$-martingale

$$m_t^X = \mathbb{E}\left[\int_0^{T} \beta_s^{-1} X_s dF_s \middle| \mathcal{F}_t\right]$$

Using the SDE for the last discounted defaultable zero-coupon:

$$dZ\left(t,T_d\right)=L_{t-}dm_t^d-\exp\left(\Gamma_t\right)m_t^d dM_t$$

and keeping the same notations as for the defaultable zero-coupon market, we get:

$$\widetilde{P}_t=m_0^X+\int_0^{t\wedge\tau}\zeta_s^X\Sigma^{-1}dA_s-\int_{]0,t\wedge\tau]}\frac{\left(X_s-\widetilde{P}_{s-}\right)}{\exp\left(\Gamma_s\right)m_s^d}dZ\left(s,T_d\right)$$

for some process ζ_s^X to be specified later on. This equation gives the decomposition of the price process of the payment at hit in terms of defaultable zero-coupon bonds, and allows to conclude on the hedging strategy for a payment at hit.

Proposition 21. *If $\widetilde{P}_t$ is the value of the payment at hit, then the hedging strategy is given by a portfolio of defaultable zero-coupon bonds and cash account made of:*

1. *a portfolio of defaultable zero-coupon bonds given by $\zeta_t^X\Sigma^{-1}A_t$*
2. *an additional position in defaultable zero-coupon of maturity T_d, given by $\frac{\left(\widetilde{P}_s-X_s\right)}{\overline{P}(t,T_d)}$*
3. *a cash amount a_t such that*

$$\zeta_t^X\Sigma^{-1}A_t+a_t\beta_t=X_t$$

where ζ_t^X is given by:

$$\zeta_t^X=\phi_t^X+\phi_t^d$$

with:

- *$\phi_t^X=\exp\left(\Gamma_t\right)\psi_t^X$, where ψ_t^X is defined through the application of the martingale representation theorem (ref) to $m_t^X=\mathbb{E}\left[\int_0^T\beta_s^{-1}X_s dF_s\middle|\mathcal{F}_t\right]$*
- *$\phi_t^d=\frac{\exp(-\Gamma_t)L_{t-}}{m_t^d}\psi_t^d$, where ψ_t^d is defined through the application of the martingale representation theorem (ref) to m_t^d*

5.2. Example: Hedging of CDS. In this section, we take once again the case of a Credit Default Swap, in order to give indications about the hedging strategy for a standard market contract. As described in the previous section, the market CDS contract definition involves a payment at default, so that the hedging portfolio will be slightly different than in the case of a discrete tenor CDS.

We proceed as follows: assuming independence between credit and interest rates, we give an expression of the market CDS contract using zero-coupon bonds and defaultable zero-coupon bonds. As for the discrete tenor case, we consider only the redemption payment, which is given as:

$$\begin{aligned}RP_T\left(t\right)&=\beta_t\mathbb{E}^{\mathbb{Q}^*}\left[\beta_\tau^{-1}\mathbf{1}_{\{t<\tau\leq T\}}\left|\mathcal{G}_t\right.\right]\\&=\mathbf{1}_{\{\tau>t\}}\beta_t\exp\left(\Gamma_t\right)\mathbb{E}^{\mathbb{Q}^*}\left[\int_t^T\beta_u^{-1}dF_u\left|\mathcal{F}_t\right.\right]\end{aligned}$$

Then, applying an integration by part formula, we get:

$$\begin{aligned}\beta_t^{-1}dF_t &= \beta_t^{-1}F_t - F_t d\beta_t^{-1}\\ &= \beta_t^{-1}F_t + r_t\beta_t^{-1}F_t dt\end{aligned}$$

and using the independence hypothesis,

$$\begin{aligned}RP_T(t) &= \mathbf{1}_{\{\tau>t\}}\beta_t\exp(\Gamma_t)\left\{\mathbb{E}^{\mathbb{Q}^*}\left[\left[\beta_s^{-1}F_s\right]_t^T|\mathcal{F}_t\right]+\mathbb{E}^{\mathbb{Q}^*}\left[\int_t^T r_s\beta_s^{-1}F_s ds\,|\mathcal{F}_t\right]\right\}\\
&= \mathbf{1}_{\{\tau>t\}}\beta_t\exp(\Gamma_t)\left\{\mathbb{E}^{\mathbb{Q}^*}\left[\beta_T^{-1}F_T-\beta_t^{-1}F_t\,|\mathcal{F}_t\right]+\mathbb{E}^{\mathbb{Q}^*}\left[\int_t^T r_s\beta_s^{-1}F_s ds\,|\mathcal{F}_t\right]\right\}\\
&= \mathbf{1}_{\{\tau>t\}}P(t,T)\exp(\Gamma_t)-\overline{P}(t,T)+\mathbf{1}_{\{\tau>t\}}-\mathbf{1}_{\{\tau>t\}}\exp(\Gamma_t)\\
&\quad -\mathbf{1}_{\{\tau>t\}}\beta_t\exp(\Gamma_t)\mathbb{E}^{\mathbb{Q}^*}\left[\int_t^T r_s\beta_s^{-1}(1-F_s)\,ds\,|\mathcal{F}_t\right]\\
&\quad +\mathbf{1}_{\{\tau>t\}}\beta_t\exp(\Gamma_t)\mathbb{E}^{\mathbb{Q}^*}\left[\int_t^T r_s\beta_s^{-1}ds\,|\mathcal{F}_t\right]\\
&= \mathbf{1}_{\{\tau>t\}}-\overline{P}(t,T)-\mathbf{1}_{\{\tau>t\}}\beta_t\exp(\Gamma_t)\mathbb{E}^{\mathbb{Q}^*}\left[\int_t^T r_s\beta_s^{-1}(1-F_s)\,ds\,|\mathcal{F}_t\right]\end{aligned}$$

The price at time t of the defaultable zero-coupon bond of maturity s is given by:

$$\overline{P}(t,s)=\mathbf{1}_{\{\tau>t\}}\beta_t\exp(\Gamma_t)\mathbb{E}^{\mathbb{Q}^*}\left[\beta_s^{-1}(1-F_s)\,|\mathcal{F}_t\right]$$

We recall that:

$$\begin{aligned}\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=s} &= \left.\frac{\partial}{\partial T}\left\{\mathbb{E}^{\mathbb{Q}^*}\left[\beta_t\beta_T^{-1}\,|\mathcal{F}_t\right]\right\}\right|_{T=s}=\mathbb{E}^{\mathbb{Q}^*}\left[\left.\frac{\partial}{\partial T}\left(\beta_t\beta_T^{-1}\right)\right|_{T=s}|\mathcal{F}_t\right]\\
&= \mathbb{E}^{\mathbb{Q}^*}\left[r_s\beta_t\beta_s^{-1}\,|\mathcal{F}_t\right]\end{aligned}$$

Denoting by $A_t=\mathbf{1}_{\{\tau>t\}}\beta_t\exp(\Gamma_t)\mathbb{E}^{\mathbb{Q}^*}\left[\int_t^T r_s\beta_s^{-1}(1-F_s)\,ds\,|\mathcal{F}_t\right]$, and using the independence property, we get:

$$\begin{aligned}A_t &= \int_t^T\mathbf{1}_{\{\tau>t\}}\beta_t\exp(\Gamma_t)\mathbb{E}^{\mathbb{Q}^*}\left[r_s\beta_s^{-1}\,|\mathcal{F}_t\right]\mathbb{E}^{\mathbb{Q}^*}\left[1-F_s\,|\mathcal{F}_t\right]ds\\
&= \int_t^T\mathbf{1}_{\{\tau>t\}}\beta_t\exp(\Gamma_t)P(t,s)^{-1}\mathbb{E}^{\mathbb{Q}^*}\left[\beta_t\beta_s^{-1}\,|\mathcal{F}_t\right]\mathbb{E}^{\mathbb{Q}^*}\left[r_s\beta_s^{-1}\,|\mathcal{F}_t\right]\mathbb{E}^{\mathbb{Q}^*}\left[1-F_s\,|\mathcal{F}_t\right]ds\\
&= \int_t^T\mathbf{1}_{\{\tau>t\}}\beta_t\exp(\Gamma_t)P(t,s)^{-1}\mathbb{E}^{\mathbb{Q}^*}\left[r_s\beta_s^{-1}\,|\mathcal{F}_t\right]\mathbb{E}^{\mathbb{Q}^*}\left[\beta_t\beta_s^{-1}(1-F_s)\,|\mathcal{F}_t\right]ds\end{aligned}$$

so that finally

$$A_t=\int_t^T P(t,s)^{-1}\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=s}\overline{P}(t,s)\,ds$$

The redemption payment of a market CDS contract reads:

$$RP_T(t) = \mathbf{1}_{\{\tau > t\}} - \overline{P}(t,T) - \int_t^T P(t,s)^{-1} \left. \frac{\partial P(t,T)}{\partial T} \right|_{T=s} \overline{P}(t,s)\, ds \qquad (12)$$

This expression does not gives in itself a replication strategy, but it allows to extend the result presented for the discrete tenor case, in so far as, under the hypothesis of deterministic interest rates, the market CDS can be seen as a limit case the discrete tenor. Indeed, (12) becomes the continuous version of (11), when $\max_k \{|T_{k+1} - T_k|\} \to 0$.

6. Conclusion

We have shown that a market consisting of defaultable zero-coupon bonds is complete, and that, in the case of an information flow modelled through a d-dimensional Brownian motion, the hedging strategy involves $d+1$ defaultable zero-coupon bonds and the cash account. The hedging strategy is still difficult to interpret as a concrete trading strategy, apart from some specific cases. It remains to study the hedging in a credit market composed of several issuers, with correlated default. This is still an open question.

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